SEMICLASSICAL LIMITS OF QUANTIZED COORDINATE RINGS

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Dedicated to S. K. Jain on the occasion of his 70th birthday

Abstract. This paper offers an expository account of some ideas, methods, and conjectures concerning quantized coordinate rings and their semiclassical limits, with a particular focus on primitive ideal spaces. The semiclassical limit of a family of quantized coordinate rings of an affine algebraic variety V consists of the classical coordinate ring $\mathcal{O}(V)$ equipped with an associated Poisson structure. Conjectured relationships between primitive ideals of a generic quantized coordinate ring A and symplectic leaves in V (relative to a semiclassical limit Poisson structure on $\mathcal{O}(V)$) are discussed, as are breakdowns in the connections when the symplectic leaves are not algebraic. This prompts replacement of the differential-geometric concept of symplectic leaves with the algebraic concept of symplectic cores, and a reformulated conjecture is proposed: The primitive spectrum of A should be homeomorphic to the space of symplectic cores in V, and to the Poisson-primitive spectrum of $\mathcal{O}(V)$. Various examples, including both quantized coordinate rings and enveloping algebras of solvable Lie algebras, are analyzed to support the choice of symplectic cores to replace symplectic leaves.

0. Introduction

By now, the "Cheshire cat" description of quantum groups is well known – a quantum group is not a group at all, but something that remains when a group has faded away, leaving an algebra of functions behind. The appropriate functions depend on which category of group is under investigation. We concentrate here on (affine) algebraic groups G, on which the natural functions of interest are the polynomial functions. These constitute the classical coordinate ring of G, which we denote $\mathcal{O}(G)$. (The group structure on G induces a Hopf algebra structure on $\mathcal{O}(G)$, but we shall not make use of that.) A *quantized coordinate ring* of G is, informally, a deformation of $\mathcal{O}(G)$, in the sense that it is an algebra with a set of generators patterned after those in $\mathcal{O}(G)$, but with a new multiplication that is typically noncommutative. Examples and references will be given in Section 1. We do not address the question of what properties are required to qualify an algebra as a quantized coordinate ring – this remains a fundamental open problem. Quantized coordinate rings have also been defined for a number of algebraic varieties other than algebraic groups, and our discussion will incorporate them as well.

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Many parallels have been found between the structures of quantized and classical coordinate rings, and general principles for organizing and predicting such parallels are needed. The present paper concentrates on a circle of ideas and results focussed on ideal structure, particularly spaces of prime or primitive ideals. The theme/principle we follow, based on much previous work, can be stated this way:

- *The primitive ideals of a suitably generic quantized coordinate ring of an algebraic variety V should match subsets of V in some partition defined through the geometry of V and a Poisson structure obtained from a semiclassical limit process.*

Many of the terms just mentioned require explanations, which we will give over the course of the paper. Here we just mention that, in the above statement, "generic" refers to the assumption that suitable parameters in the construction of the quantized coordinate ring should be non-roots of unity.

To begin the story (omitting many definitions and details), we refer to the results of Soibelman and Vaksman [**51**, **45**, **46**], who studied the "standard" generic quantized coordinate rings of simple compact Lie groups K. They established a bijection between the irreducible *-representations of K (on Hilbert spaces) and the symplectic leaves in K (relative to a Poisson structure arising from the quantization). This amounts to a linkage between primitive ideals and symplectic leaves, a relationship which is a key ingredient of the Orbit Method from Lie theory. Informed by this principle, and inspired by the work of Soibelman and Vaksman, Hodges and Levasseur conjectured that similar bijections should exist for semisimple complex algebraic groups [**22**]. The case of $SL_2(\mathbb{C})$ being easy [**22**, Appendix], they first verified the conjecture for $SL_3(\mathbb{C})$ [op. cit.], and then for $SL_n(\mathbb{C})$ [**23**]. In later work with Toro [**24**], they verified it for connected semisimple groups. In light of these achievements, it is natural to pose this conjecture for other classes of generic quantized coordinate rings. (It is easily seen that the above conjecture cannot hold for non-generic quantized coordinate rings. In such cases, the quantized coordinate rings are usually finitely generated modules over their centers, and they have far more primitive ideals than can be matched to symplectic leaves.)

In the specific cases just mentioned, the symplectic leaves turn out to be algebraic, in the sense that they are locally closed in the Zariski topology. Hodges, Levasseur, and Toro pointed out in [**24**] that symplectic leaves need not be algebraic for Poisson structures arising from multiparameter quantizations, and that the above conjecture cannot be expected to hold in such cases. We argue that it should not be surprising that the concept of symplectic leaves, which comes from differential geometry, is not always well suited for algebraic problems. Thus, symplectic leaves should be replaced by more algebraically defined objects. The notion of symplectic cores introduced by Brown and Gordon [**6**] fills the role well, up to the present state of knowledge; we will give evidence to buttress this statement.

Our aim here is to present an account of the above story, with introductions to and discussions of the relevant concepts. In particular, the tour will pass through way stations such as *quantized coordinate rings*, *semiclassical limits*, *Poisson structures*, *symplectic leaves*, the *Orbit Method*, *symplectic cores*, and the *Dixmier map*. By the end of the tour,

we will be in purely algebraic territory, where we can formulate a conjecture that does not require any differential geometry (i.e., symplectic leaves). Namely:

- *If A is a generic quantized coordinate ring of an affine algebraic variety V over an algebraically closed field of characteristic zero, and if V is given the Poisson structure arising from an appropriate semiclassical limit, then the spaces of primitive ideals in A and symplectic cores in V, with their respective Zariski topologies, are homeomorphic.*

A parallel conjecture relates the prime and primitive spectra of A to the spaces of Poisson prime and Poisson-primitive ideals in $\mathcal{O}(V)$.

Fix a base field k throughout the paper; all algebras mentioned will be unital k-algebras. This field can be general at first, but then we will require it to have characteristic zero, and/or be algebraically closed. When discussing symplectic leaves, we restrict k to $\mathbb{R}$ or $\mathbb{C}$.

1. Quantized coordinate rings

We begin by recalling two basic examples, to clarify the idea that a quantized coordinate ring of an algebraic group (or variety) is, loosely speaking, a deformation of the classical coordinate ring. References to many other examples are given in §§1.2, 1.4, 1.5.

1.1. Quantum SL_2. Recall that the group $SL_2(k)$ is a closed subvariety of the variety of 2×2 matrices over k, defined by the single equation "determinant = 1". The coordinate ring of the matrix variety is naturally realized as a polynomial ring in four variables X_{ij}, corresponding to the functions that pick out the four entries of the matrices. The coordinate ring of $SL_2(k)$ can thus be described as follows:

$$\mathcal{O}(SL_2(k)) = k[X_{11}, X_{12}, X_{21}, X_{22}]/\langle X_{11}X_{22} - X_{12}X_{21} - 1\rangle.$$

To "quantize" this coordinate ring, we replace the commutative multiplication by a non-commutative one, parametrized by a nonzero scalar q, as below. The reasons for this particular choice of relations will not be given here; see [**4**, §§I.1.6, I.1.8], for instance, for a discussion.

Given a choice of scalar $q \in k^\times$, the "standard" one-parameter *quantized coordinate ring of* $SL_2(k)$ is the k-algebra $\mathcal{O}_q(SL_2(k))$ presented by generators X_{11}, X_{12}, X_{21}, X_{22} and the following relations:

$$\begin{aligned}
X_{11}X_{12} &= qX_{12}X_{11} & X_{11}X_{21} &= qX_{21}X_{11} \\
X_{12}X_{22} &= qX_{22}X_{12} & X_{21}X_{22} &= qX_{22}X_{21} \\
X_{12}X_{21} &= X_{21}X_{12} & X_{11}X_{22} - X_{22}X_{11} &= (q - q^{-1})X_{12}X_{21} \\
X_{11}X_{22} - qX_{12}X_{21} &= 1\,.
\end{aligned}$$

The case when $q = 1$ is special: The first six relations then reduce to saying that the generators X_{ij} commute with each other, the last reduces to the defining relation for the

variety $SL_2(k)$, and so the algebra $\mathcal{O}_1(SL_2(k))$ is just the classical coordinate ring. We write this, very informally, as

$$\mathcal{O}(SL_2(k)) = \lim_{q\to 1} \mathcal{O}_q(SL_2(k));$$

it is our first example of a "semiclassical limit".

1.2. Quantum matrices, quantum SL_n and GL_n. The pattern indicated in §1.1 extends to definitions of "standard" single parameter quantized coordinate rings $\mathcal{O}_q(M_n(k))$, $\mathcal{O}_q(SL_n(k))$, and $\mathcal{O}_q(GL_n(k))$ for all positive integers n. Multiparameter versions, which we label in the form $\mathcal{O}_{\lambda,\boldsymbol{p}}(-)$, have also been defined. Generators and relations for these algebras may be found, for instance, in [**12**, §§1.2–1.4; **4**, §§I.2.2–I.2.4].

1.3. Quantum affine spaces. The coordinate ring of affine n-space over k is the polynomial algebra in n indeterminates, and the most basic quantization is obtained by replacing commutativity ($xy = yx$) with *q-commutativity*: $xy = qyx$. Thus, the "standard" one-parameter *quantized coordinate ring of k^n*, relative to a choice of scalar $q \in k^\times$, is the k-algebra

$$\mathcal{O}_q(k^n) = k\langle x_1, \dots, x_n \mid x_i x_j = q x_j x_i \text{ for } 1 \le i < j \le n\rangle.$$

The multiparameter version of this algebra requires an $n \times n$ matrix of nonzero scalars, $\boldsymbol{q} = (q_{ij})$, which is *multiplicatively antisymmetric* in the sense that $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for all i, j. The *multiparameter quantized coordinate ring of k^n* corresponding to a choice of $\boldsymbol{q}$ is the k-algebra

$$\mathcal{O}_{\boldsymbol{q}}(k^n) = k\langle x_1, \dots, x_n \mid x_i x_j = q_{ij} x_j x_i \text{ for all } i, j\rangle.$$

In the one-parameter case, we can write $\mathcal{O}(k^n) = \lim_{q\to 1} \mathcal{O}_q(k^n)$ in the same sense as above. For the multiparameter case, we imagine a limit in which all $q_{ij} \to 1$.

1.4. Quantized coordinate rings of semisimple groups. The single parameter versions of these Hopf algebras, which we denote $\mathcal{O}_q(G)$, were first defined for semisimple algebraic groups G of classical type (types A, B, C, D) via generators and relations, by Faddeev, Reshetikhin, and Takhtadjan [**43**] and Takeuchi [**48**]. A detailed development (done for $k = \mathbb{C}$, but the pattern is the same over other fields) can be found in [**32**, Chapter 9]. In most of the more recent literature, $\mathcal{O}_q(G)$ is defined as a restricted Hopf dual of the quantized enveloping algebra of the Lie algebra of G (e.g., see [**4**, Chapter I.7]). This is a more uniform approach, which also covers groups of exceptional type. That the two approaches yield the same Hopf algebras in the classical cases was established by Hayashi [**20**] and Takeuchi [**48**] (see [**32**, Theorem 11.22]).

The single parameter algebras $\mathcal{O}_q(G)$ constitute the "standard" quantized coordinate rings of semisimple groups. Multiparameter versions, which we label $\mathcal{O}_{q,p}(G)$, were introduced by Hodges, Levasseur, and Toro [**24**].

1.5. Additional examples. Quantized coordinate rings, both single- and multiparameter, have been defined for many algebraic varieties, such as algebraic tori, toric varieties, and versions of affine spaces related to classical groups of types B, C, D. For a general survey, see [**12**, Section 1]. Quantized toric varieties were introduced in [**26**] (see also [**17**, **16**]). A family of iterated skew polynomial algebras covering multiparameter quantized euclidean and symplectic spaces was introduced by Oh [**38**] and extended by Horton [**25**] (see also [**14**, §2.5] for the odd-dimensional euclidean case). Among other algebras that have been studied in the literature, we mention quantized coordinate rings for varieties of antisymmetric matrices [**47**] and varieties of symmetric matrices [**37**, **28**].

1.6. Limits of families of algebras. The semiclassical limits informally introduced in §§1.1, 1.3 are more properly viewed in the framework of families of algebras. For example, the algebras $\mathcal{O}_q(SL_2(k))$ are quotients of a single algebra over a Laurent polynomial ring $k[t^{\pm 1}]$, namely the algebra A given by generators X_{11}, X_{12}, X_{21}, X_{22} and relations as in §1.1, but with q replaced by t:

$$\begin{aligned} &X_{11}X_{12} = tX_{12}X_{11} && X_{11}X_{21} = tX_{21}X_{11} \\ &X_{12}X_{22} = tX_{22}X_{12} && X_{21}X_{22} = tX_{22}X_{21} \\ &X_{12}X_{21} = X_{21}X_{12} && X_{11}X_{22} - X_{22}X_{11} = (t - t^{-1})X_{12}X_{21} \\ &X_{11}X_{22} - tX_{12}X_{21} = 1\,. \end{aligned} \tag{1.6a}$$

For each $q \in k^\times$, there is a natural identification $A/(t-q)A \equiv \mathcal{O}_q(SL_2(k))$. The "limit as $q \to 1$" is then simply the case $q = 1$ of these identifications: $A/(t-1)A \equiv \mathcal{O}(SL_2(k))$.

Similarly, if we take

$$B = k[t^{\pm 1}]\langle x_1, \dots, x_n \mid x_i x_j = t x_j x_i \text{ for } 1 \le i < j \le n\rangle, \tag{1.6b}$$

then $B/(t-q)B \equiv \mathcal{O}_q(k^n)$ for all $q \in k^\times$, and

$$\lim_{q \to 1} \mathcal{O}_q(k^n) = B/(t-1)B \equiv \mathcal{O}(k^n).$$

The multiparameter algebras $\mathcal{O}_{\boldsymbol{q}}(k^n)$ can, likewise, be set up as common quotients of an algebra over a Laurent polynomial ring $k[t_{ij}^{-1} \mid 1 \le i < j \le n]$. However, for purposes such as obtaining Poisson structures on semiclassical limits, we need to be able to exhibit the $\mathcal{O}_{\boldsymbol{q}}(k^n)$ as quotients of $k[t^{\pm 1}]$-algebras. There are many ways to do this; we will discuss some in §2.3.

1.7. An older example: the Weyl algebra. Weyl defined the algebra we now call the *first Weyl algebra* as

$$\mathbb{C}\langle x, y \mid xy - yx = \hbar i\rangle,$$

where $\hbar$ is Planck's constant and $i = \sqrt{-1}$. Physicists often use the term "classical limit" to denote the transition from a quantum mechanical system to a classical one by letting Planck's constant go to zero. The fact that $\lim\limits_{\hbar \to 0}$ of the above algebra is the polynomial ring $\mathbb{C}[x, y]$ is one instance of this point of view.

To relate this semiclassical limit to the ones above, take $k = \mathbb{C}$ and take the scalar q in quantized coordinate rings to be $e^{\hbar}$. Then $\hbar \to 0$ corresponds to $q \to 1$. In many constructions, particularly the C*-algebra quantum groups corresponding to compact Lie groups, the parameter q is either written directly in the form $e^{\hbar}$ or is taken to be a nonnegative real number, with calculations involving $e^{\hbar}$ used for motivation.

2. Semiclassical limit constructions

In the context of quantized coordinate rings, semiclassical limits are constructed via quotients of algebras over Laurent polynomial rings, as in §1.6. A different version, using associated graded rings, is needed in other arenas, particularly for enveloping algebras of Lie algebras. We describe both constructions in this section.

2.1. Semiclassical limits: commutative fibre version. Let $k[h]$ be a polynomial algebra, with the indeterminate named h as a reminder of Planck's constant. Suppose that A is a torsionfree $k[h]$-algebra, and that A/hA is commutative. Since A is then a flat $k[h]$-module, the family of factor algebras $\big(A/(h-\alpha)A\big)_{\alpha\in k}$ (or, for short, A itself) is called a *flat family* of k-algebras, and A/hA is viewed as the "limit" of the family. It may happen that some of the algebras $A/(h-\alpha)A$ collapse to zero or are otherwise not desirable. If so, it is natural to treat A as an algebra over a localization of $k[h]$ (cf. Example 2.2(c), for instance). We will usually not do this explicitly.

An immediate question is, what kind of information about the algebras $A/(h-\alpha)A$ is contained in this limit? Observe that, because of the commutativity of A/hA, all additive commutators $[a,b] = ab - ba$ in A are divisible by h. Moreover, division by h is unique, since A is torsionfree as a $k[h]$-module. Hence, we obtain a well defined binary operation $\frac{1}{h}[-,-]$ on A. This operation enjoys four key properties:

(1) Bilinearity;
(2) Antisymmetry;
(3) The Jacobi identity (thus A, equipped with $\frac{1}{h}[-,-]$, is a Lie algebra over k);
(4) The *Leibniz identities*, that is, the product rule (for derivatives) in each variable: $\frac{1}{h}[a,bc] = \big(\frac{1}{h}[a,b]\big)c + b\big(\frac{1}{h}[a,c]\big)$ for all $a,b,c \in A$, and similarly for $\frac{1}{h}[bc,a]$.

Operations satisfying properties (1)–(4) are called *Poisson brackets*.

The above Poisson bracket on A induces, uniquely, a Poisson bracket on A/hA, which we denote $\{-,-\}$. Thus, writing overbars to denote cosets modulo hA, we have

$$\{\overline{a},\overline{b}\} = \overline{\tfrac{1}{h}[a,b]}$$

for $a,b \in A$. The commutative algebra A/hA, equipped with this Poisson bracket, is called the *semiclassical limit* of the family $\big(A/(h-\alpha)A\big)_{\alpha\in k}$. Loosely speaking, the Poisson bracket on the semiclassical limit records a "first-order impression" of the commutators in A and in the algebras $A/(h-\alpha)A$.

2.2. Examples. (a) Fit the one-parameter quantum affine spaces $\mathcal{O}_q(k^n)$ into the $k[t^{\pm1}]$-algebra B of (1.6b), and set $h = t-1$. Then B represents a flat family of $k[h]$-algebras, with B/hB commutative. We identify B/hB with the polynomial ring $k[x_1,\dots,x_n]$ and

compute the resulting Poisson bracket on the indeterminates as follows. For $1 \le i < j \le n$, we have $[x_i, x_j] = hx_jx_i$ in B, and hence

$$\{x_i, x_j\} = x_ix_j$$

in $k[x_1, \dots, x_n]$. Because of the Leibniz identities, the above information determines this Poisson bracket uniquely. It may be described in full as follows:

$$\{f, g\} = \sum_{1 \le i < j \le n} x_ix_j \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial x_j} \right)$$

for all $f, g \in k[x_1, \dots, x_n]$.

(b) Take $A = k[h]\langle x, y \mid xy - yx = h \rangle$. Then $A/(h - \alpha)A \cong A_1(k)$ for all nonzero $\alpha \in k$, while A/hA can be identified with the polynomial ring $k[x, y]$. In this case, the semiclassical limit Poisson bracket on $k[x, y]$ satisfies (and is determined by)

$$\{x, y\} = 1.$$

(c) The family $\big(\mathcal{O}_q(SL_2(k))\big)_{q \in k^\times}$ fits into the $k[t^{\pm 1}]$-algebra A with generators X_{11}, X_{12}, X_{21}, X_{22} and relations (1.6a). This is a flat family over $k[h]$, where $h = t - 1$. Since t is invertible in A, the specialization $A/(h+1)A$ is zero, corresponding to the fact that A is actually a torsionfree (even free) algebra over the localization $k[h][(h+1)^{-1}]$. Here the semiclassical limit is the classical coordinate ring $\mathcal{O}(SL_2(k))$, equipped with the Poisson bracket satisfying

$$\begin{aligned} \{X_{11}, X_{12}\} &= X_{11}X_{12} & \qquad \{X_{11}, X_{21}\} &= X_{11}X_{21} \\ \{X_{12}, X_{22}\} &= X_{12}X_{22} & \{X_{21}, X_{22}\} &= X_{21}X_{22} \\ \{X_{12}, X_{21}\} &= 0 & \{X_{11}, X_{22}\} &= 2X_{12}X_{21}\,. \end{aligned}$$

2.3. Multiparameter examples. To obtain a semiclassical limit – with Poisson bracket – for a multiparameter family of algebras, we convert to a single parameter family and apply the construction of §2.1. The procedure is clear when the parameters involved are integer powers of a single parameter. For example, consider the algebras $\mathcal{O}_{\boldsymbol{q}}(k^n)$ where $\boldsymbol{q} = (q^{a_{ij}})$ for $q \in k^\times$ and an antisymmetric integer matrix (a_{ij}). Then define

$$A = k[t^{\pm 1}]\langle x_1, \dots, x_n \mid x_ix_j = t^{a_{ij}}x_jx_i \text{ for all } i, j \rangle,$$

which is a torsionfree $k[t-1]$-algebra with $A/(t-1)A$ commutative. The semiclassical limit is the polynomial algebra $k[x_1, \dots, x_n]$, equipped with the Poisson bracket satisfying

$$\{x_i, x_j\} = a_{ij}x_ix_j$$

for all i, j.

More general parameters can be dealt with by various means. A simple but ad hoc method to handle any $\mathcal{O}_{\boldsymbol{q}}(k^n)$ is via the algebra

$$A = k[h]\langle x_1, \dots, x_n \mid x_i x_j = \big(1 + (q_{ij} - 1)h\big)x_j x_i \text{ for } 1 \le i < j \le n\rangle,$$

which is set up so that $A/(h-1)A \cong \mathcal{O}_{\boldsymbol{q}}(k^n)$ and $A/hA \cong k[x_1, \dots, x_n]$. This yields a Poisson bracket satisfying $\{x_i, x_j\} = (q_{ij} - 1)x_i x_j$ for all i, j.

A variant of the previous procedure, involving quadratic rather than linear polynomials in h, is used in [**18**] to construct semiclassical limits for which the conjecture sketched in the Introduction applies to the generic multiparameter quantum affine spaces $\mathcal{O}_{\boldsymbol{q}}(k^n)$.

Inverse to the construction of semiclassical limits is the problem of *quantization*: trying to represent a given algebra supporting a Poisson bracket as a semiclassical limit of a suitable family of algebras. We will not discuss this problem except to indicate a solution for the case of homogeneous quadratic Poisson brackets on polynomial rings. Namely, suppose we have a polynomial algebra $k[x_1, \dots, x_n]$, equipped with a Poisson bracket such that $\{x_i, x_j\} = \alpha_{ij} x_i x_j$ for all i, j, where (α_{ij}) is an antisymmetric matrix of scalars over k. In place of ad hoc procedures such as the one sketched above, it is natural, assuming that $\operatorname{char} k = 0$, to use power series. In this case, set $\exp(\alpha_{ij}) = \sum_{n=0}^{\infty} \frac{1}{n!} \alpha_{ij}^n h^n \in k[[h]]$ for all i, j, and form the $k[[h]]$-algebra

$$A = k[[h]]\langle x_1, \dots, x_n \mid x_i x_j = \exp(\alpha_{ij}) x_j x_i \text{ for all } i, j\rangle.$$

The semiclassical limit algebra is $k[x_1, \dots, x_n]$, and its Poisson bracket is the original one.

Analogous $k[[h]]$-algebra constructions are given for commonly studied families of quantized coordinate rings of skew polynomial type in [**14**, Section 2].

2.4. Semiclassical limits: filtered/graded version. Suppose that A is a $\mathbb{Z}$-filtered k-algebra, say with filtration $(A_n)_{n\in\mathbb{Z}}$. Thus, the A_n are k-subspaces of A, with $A_m \subseteq A_n$ when $m \le n$, such that $A_m A_n \subseteq A_{m+n}$ for all m, n. We will assume that the filtration is *exhaustive*, that is, that $\bigcup_{n\in\mathbb{Z}} A_n = A$. Note that we must have $1 \in A_0$; thus, A_0 is a unital subalgebra of A. Finally, let $\operatorname{gr} A = \bigoplus_{n\in\mathbb{Z}} \operatorname{gr}_n A$ be the associated graded algebra, where $\operatorname{gr}_n A = A_n/A_{n-1}$.

Now assume that $\operatorname{gr} A$ is commutative. Homogeneous elements $a \in \operatorname{gr}_m A$ and $b \in \operatorname{gr}_n A$ can be lifted to elements $\widehat{a} \in A_m$ and $\widehat{b} \in A_n$, and since $\operatorname{gr} A$ is commutative, the commutator $[\widehat{a}, \widehat{b}]$ must lie in A_{m+n-1}. We then set $\{a, b\}$ equal to the coset of $[\widehat{a}, \widehat{b}]$ in $\operatorname{gr}_{m+n-1} A$. It is an easy exercise, left to the reader, to verify that $\{a, b\}$ is well defined, and that the extension of $\{-,-\}$ to sums of homogeneous elements defines a Poisson bracket on $\operatorname{gr} A$. The commutative algebra $\operatorname{gr} A$, equipped with this Poisson bracket, is called the *semiclassical limit* of A.

More generally, assume there is an integer $d < 0$ such that $[A_m, A_n] \subseteq A_{m+n+d}$ for all $m, n \in \mathbb{Z}$. This assumption of course forces $\operatorname{gr} A$ to be commutative. Modify the definition above by setting $\{a, b\}$ equal to the coset of $[\widehat{a}, \widehat{b}]$ in $\operatorname{gr}_{m+n+d} A$, for $a \in \operatorname{gr}_m A$ and $b \in \operatorname{gr}_n A$. This recipe again produces a well defined Poisson bracket on $\operatorname{gr} A$ [**34**, Lemma 2.7].

2.5. Bridging the two constructions. The semiclassical limit of a $\mathbb{Z}$-filtered algebra A constructed in §2.4 can also be obtained by applying the construction of §2.1 to an auxiliary algebra, namely the *Rees ring*

$$\widetilde{A} := \sum_{n\in\mathbb{Z}} A_n h^n \subseteq A[h^{\pm 1}],$$

where $A[h^{\pm 1}]$ is a Laurent polynomial ring over A. Since $1 \in A_0$, the polynomial algebra $k[h]$ is a subalgebra of A, and we note that $\widetilde{A}$ is a torsionfree $k[h]$-algebra. (It is not a $k[h^{\pm 1}]$-algebra unless $A_{-1} = A_0$, in which case all $A_n = A_0$.) On one hand, $\widetilde{A}/(h-1)\widetilde{A} \cong A$. On the other, $\widetilde{A}/h\widetilde{A} \cong \operatorname{gr} A$, because $h\widetilde{A} = \sum_{n\in\mathbb{Z}} A_{n-1}h^n$. Thus, if $\operatorname{gr} A$ is commutative, we have a Poisson bracket $\frac{1}{h}[-,-]$ on $\widetilde{A}$, which induces a Poisson bracket $\{-,-\}_1$ on $\operatorname{gr} A$ as in §2.1. This bracket concides with the Poisson bracket $\{-,-\}_4$ constructed in §2.4, as follows.

Start with $a \in \operatorname{gr}_m A$ and $b \in \operatorname{gr}_n A$, and lift these elements to $\widehat{a} \in A_m$ and $\widehat{b} \in A_n$. With respect to the natural epimorphism $\pi : \widetilde{A} \to \operatorname{gr} A$, the elements a and b lift to $\widehat{a}h^m, \widehat{b}h^n \in \widetilde{A}$. Hence,

$$\{a,b\}_1 = \pi\big(\frac{1}{h}[\widehat{a}h^m, \widehat{b}h^n]\big) = \pi([\widehat{a},\widehat{b}]h^{m+n-1}) = [\widehat{a},\widehat{b}] + A_{m+n-2} = \{a,b\}_4\,.$$

Therefore $\{-,-\}_1 = \{-,-\}_4$.

2.6. Example: enveloping algebras. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over k, and put the standard (nonnegative) filtration on the enveloping algebra $U(\mathfrak{g})$, so that $U(\mathfrak{g})_0 = k$ and $U(\mathfrak{g})_1 = k + \mathfrak{g}$, while $U(\mathfrak{g})_n = U(\mathfrak{g})_1^n$ for $n > 1$. The associated graded algebra is commutative, and is naturally identified with the symmetric algebra $S(\mathfrak{g})$ of the vector space $\mathfrak{g}$. In particular, we use the same symbol to denote an element of $\mathfrak{g}$ and its coset in $\operatorname{gr}_1 U(\mathfrak{g}) = S(\mathfrak{g})_1$. Then $S(\mathfrak{g})$ is the semiclassical limit of $U(\mathfrak{g})$, equipped with the Poisson bracket satisfying

$$\{e,f\} = [e,f]$$

for all $e, f \in \mathfrak{g}$, where $[e,f]$ denotes the Lie product in $\mathfrak{g}$. The above formula determines $\{-,-\}$ uniquely, since $\mathfrak{g}$ generates $S(\mathfrak{g})$.

Now view the dual space $\mathfrak{g}^*$ as an algebraic variety, namely the affine space $\mathbb{A}^{\dim \mathfrak{g}}$. The coordinate ring $\mathcal{O}(\mathfrak{g}^*)$ is a polynomial algebra over k in $\dim \mathfrak{g}$ indeterminates, as is $S(\mathfrak{g})$. There is a canonical isomorphism

$$\theta : S(\mathfrak{g}) \xrightarrow{\cong} \mathcal{O}(\mathfrak{g}^*) \tag{2.6}$$

which sends each $e \in \mathfrak{g}$ to the polynomial function on $\mathfrak{g}^*$ given by evaluation at e, that is, $\theta(e)(\alpha) = \alpha(e)$ for $\alpha \in \mathfrak{g}^*$. (This isomorphism is often treated as an identification of the algebras $S(\mathfrak{g})$ and $\mathcal{O}(\mathfrak{g}^*)$.) Via θ, the Poisson bracket on $S(\mathfrak{g})$ obtained from the semiclassical limit process above carries over to a Poisson bracket on $\mathcal{O}(\mathfrak{g}^*)$, known as the *Kirillov-Kostant-Souriau Poisson bracket*.

If $\{e_1, \dots, e_n\}$ is a basis for $\mathfrak{g}$, then $S(\mathfrak{g}) = k[e_1, \dots, e_n]$ and θ sends the e_i to indeterminates x_i such that $\mathcal{O}(\mathfrak{g}^*) = k[x_1, \dots, x_n]$. An explicit description of the KKS Poisson bracket on $\mathcal{O}(\mathfrak{g}^*)$ can be obtained in terms of the structure constants of $\mathfrak{g}$, as follows. These constants are scalars $c_{ij}^l \in k$ such that $[e_i, e_j] = \sum_l c_{ij}^l e_l$ for all i, j. Since $\{e_i, e_j\} = [e_i, e_j]$ in $S(\mathfrak{g})$, an application of θ yields $\{x_i, x_j\} = \sum_l c_{ij}^l x_l$ for all i, j. It follows that

$$\{p, q\} = \sum_{i,j,l} c_{ij}^l x_l \frac{\partial p}{\partial x_i} \frac{\partial q}{\partial x_j}$$

for $p, q \in \mathcal{O}(\mathfrak{g}^*)$ [**7**, Proposition 1.3.18]. To see this, just check that the displayed formula determines a Poisson bracket on $\mathcal{O}(\mathfrak{g}^*)$ which agrees with the KKS bracket on pairs of indeterminates.

The KKS Poisson bracket on $\mathcal{O}(\mathfrak{g}^*)$ can also be obtained by applying the method of §2.1 to the homogenization of $U(\mathfrak{g})$, that is, the $k[h]$-algebra A with generating vector space $\mathfrak{g}$ and relations $ef - fe = h[e, f]$ for $e, f \in \mathfrak{g}$ (where $[e, f]$ again denotes the Lie product in $\mathfrak{g}$). Here $A/hA \cong S(\mathfrak{g}) \cong \mathcal{O}(\mathfrak{g}^*)$ and $A/(h - \lambda)A \cong U(\mathfrak{g})$ for all $\lambda \in k^\times$.

3. Symplectic leaves

We introduce symplectic leaves first in the context of Poisson manifolds, following the original definition of Weinstein [**54**], and then we carry the concept over to complex affine Poisson varieties, following Brown and Gordon [**6**].

3.1. Poisson algebras. We reiterate the general definition from §2.1: a *Poisson bracket* on a k-algebra R is any antisymmetric bilinear map $R \times R \to R$ which satisfies the Jacobi and Leibniz identities. Unless a special notation imposes itself, we denote all Poisson brackets by curly braces: $\{-, -\}$.

A *Poisson algebra* over k is just a k-algebra R equipped with a particular Poisson bracket. We restrict our attention to commutative Poisson algebras in the present paper. As for the noncommutative case, Farkas and Letzter have shown that Poisson brackets essentially reduce to commutators [**11**, Theorem 1.2]: If R is a prime ring which is not commutative, any Poisson bracket on R is a multiple of the commutator bracket by an element of the extended centroid of R.

3.2. Symplectic leaves in Poisson manifolds. Let M be a smooth manifold, and let $C^\infty(M)$ denote the algebra of smooth real-valued functions on M. (Some authors replace $C^\infty(M)$ by the algebra of smooth or analytic complex-valued functions.) A *Poisson structure* on M is a choice of Poisson bracket on $C^\infty(M)$, so that $C^\infty(M)$ becomes a Poisson algebra. A smooth manifold, together with a choice of Poisson structure, is called a *Poisson manifold*.

Now assume that M is a Poisson manifold. For each $f \in C^\infty(M)$, the map $X_f = \{f, -\}$ is a derivation on $C^\infty(M)$ and thus a vector field on M. Such vector fields are called *Hamiltonian vector fields* (for the given Poisson structure), and the flows (or integral curves) of Hamiltonian vector fields are known as *Hamiltonian paths*. More specifically, a smooth path $\gamma : [0, 1] \to M$ is Hamiltonian provided there is some $f \in C^\infty(M)$ such

that, at each point $\gamma(t)$ along the path, the tangent vector $d\gamma/dt$ equals $X_f|_{\gamma(t)}$. Since the change from a Hamiltonian path following the flow of a vector field X_f to one following a different vector field X_g need not be smooth, one must work with *piecewise Hamiltonian paths*, i.e., finite concatenations of Hamiltonian paths.

These paths determine an equivalence relation on M, points p and p' being equivalent if and only if there is a piecewise Hamiltonian path in M running from p to p'. The resulting equivalence classes are called *symplectic leaves*, and the partition of M as the disjoint union of its symplectic leaves is known as the *symplectic foliation* of M.

3.3. Poisson bivector fields. For many purposes, it is more useful to record a Poisson structure in the form of a bivector field rather than a Poisson bracket. In particular, this allows the most direct definition of Poisson structures on non-affine algebraic varieties.

Let M be a Poisson manifold. For a point $p \in M$, let $\mathfrak{m}_p$ denote the maximal ideal of $C^\infty(M)$ consisting of those functions that vanish at p. Evaluation of Poisson brackets at p induces an antisymmetric bilinear form π_p on the cotangent space $\mathfrak{m}_p/\mathfrak{m}_p^2$, where

$$\pi_p(f + \mathfrak{m}_p^2,\, g + \mathfrak{m}_p^2) = \{f, g\}(p)$$

for $f, g \in \mathfrak{m}_p$. Now π_p acts in each variable as a linear map in the dual space of $\mathfrak{m}_p/\mathfrak{m}_p^2$, that is, as a tangent vector to M at p. Since π_p is antisymmetric, it is thus a *tangent bivector* at p, namely an element of $T_p(M) \wedge T_p(M)$. The map $\pi : p \mapsto \pi_p$ is a smooth global section of $\Lambda^2 T_M$, that is, a *tangent bivector field* on M. To recover the Poisson bracket on $C^\infty(M)$ from the bivector field π, observe that $\{f, g\}(p) = \{f - f(p),\, g - g(p)\}(p)$ for $f, g \in C^\infty(M)$ and $p \in M$, which we rewrite in the form

$$\{f, g\}(p) = \pi_p(df(p), dg(p)), \tag{3.3}$$

where $df(p) = f - f(p) + \mathfrak{m}_p^2 \in \mathfrak{m}_p/\mathfrak{m}_p^2$ and similarly for $dg(p)$.

Conversely, via (3.3) any tangent bivector field π on M induces an antisymmetric bilinear map $\{-, -\}$ on $C^\infty(M)$ satisfying the Leibniz conditions. This is a Poisson bracket exactly when the Jacobi identity is satisfied, which is equivalent to the vanishing of the *Schouten bracket* $[\pi, \pi]$ (which we will not define here; see [**1**, p. 44; **53**, 2nd. ed., Remark 2.2(3)], for instance). A *Poisson bivector field* on M is any tangent bivector field π for which $[\pi, \pi] = 0$. As indicated in the sketch above, Poisson brackets on $C^\infty(M)$ correspond bijectively to Poisson bivector fields on M.

3.4. Poisson varieties. For any complex algebraic variety V, the definition of a *Poisson bivector field* on V can be copied from §3.3 – it is any tangent bivector field π on V for which $[\pi, \pi] = 0$. In the context of algebraic geometry, however, the map $\pi : V \to \Lambda^2 T_V$ is required to be a regular function. Now one defines a *Poisson variety* to be a complex algebraic variety equipped with a particular Poisson bivector field. Associated concepts are defined by requiring compatibility with these bivector fields. For example, a *Poisson morphism* from a Poisson variety (V, π) to a Poisson variety (W, π') is a regular map $\phi : V \to W$ such that $(T\phi \wedge T\phi)\pi = \pi'\phi$. A *Poisson subvariety* of V is a subvariety X such that the inclusion map $X \to V$ is a Poisson morphism.

If V is an affine Poisson variety, the formula (3.3) defines a Poisson bracket on $\mathcal{O}(V)$. Conversely, any Poisson bracket on $\mathcal{O}(V)$ induces a Poisson bivector field on V as in §3.3. Thus, affine Poisson varieties can equally well be defined as complex affine varieties whose coordinate rings are Poisson algebras. This point of view can be extended to arbitrary varieties by defining a Poisson variety to be a complex algebraic variety whose sheaf of regular functions is a sheaf of Poisson algebras.

3.5. Smooth Poisson varieties as manifolds. In order to define symplectic leaves in Poisson varieties, manifold structures are needed. The fundamental result is that any smooth (i.e., nonsingular) complex variety V has a unique structure as a complex analytic manifold (e.g., [**44**, Chapter II, §2.3]). This allows one to view V as a smooth manifold. If V is a Poisson variety, its chosen Poisson bivector field π is necessarily smooth (because it is regular), and so V together with π becomes a Poisson manifold. One can achieve this result with Poisson brackets as well, by showing that any Poisson bracket on $\mathcal{O}(V)$ extends uniquely to a Poisson bracket on the algebra of smooth complex functions on V; taking real parts then yields a Poisson bracket on $C^\infty(V)$.

Given a smooth Poisson variety V, we view V as a smooth manifold as above, and define the *symplectic leaves* of V to be the symplectic leaves of the manifold V, defined as in §3.2.

3.6. Symplectic leaves in singular Poisson varieties. Let V be an arbitrary complex variety, and define the sequence of closed subvarieties

$$V_0 = V \supset V_1 \supset \cdots \supset V_m = \varnothing,$$

where each V_{i+1} is the singular locus of V_i. To build this chain, recall first that the singular locus of a nonempty variety is a proper closed subvariety. Since V is a noetherian topological space, the chain must eventually reach the empty set.

If V is a Poisson variety, then V_1 is a Poisson subvariety [**42**, Corollary 2.4]. By induction, all the V_i are Poisson subvarieties of V. Consequently, V is (canonically) the disjoint union of smooth locally closed Poisson subvarieties $Z_i := V_{i-1} \setminus V_i$. Following [**6**, §3.5], we define the *symplectic leaves* of V to be the symplectic leaves of the various Z_i, defined as in §3.5.

3.7. Example. There is a known recipe, described in [**22**, Appendix A], for determining the symplectic leaves in a semisimple complex algebraic group G, relative to the Poisson structure arising from the "standard quantization" of G. For illustration, we present the case $G = SL_2(\mathbb{C})$; details are given in [**22**, Theorem B.2.1]. The Poisson bracket on $\mathcal{O}(G)$ is described in §2.2(c) above. The symplectic leaves in G are as follows:

- the singletons $\left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \right\}$, for $\alpha \in \mathbb{C}^\times$;
- the sets $\left\{ \begin{bmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{bmatrix} \;\middle|\; \alpha, \gamma \in \mathbb{C}^\times \right\}$ and $\left\{ \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix} \;\middle|\; \alpha, \beta \in \mathbb{C}^\times \right\}$;
- the sets $\left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in G \;\middle|\; \beta = \lambda\gamma \neq 0 \right\}$, for $\lambda \in \mathbb{C}^\times$.

3.8. Example. The standard example of a non-algebraic solvable Lie algebra is a 3-dimensional complex Lie algebra $\mathfrak{g}$ with basis $\{e_1, e_2, e_3\}$ such that

$$[e_1, e_2] = e_2 \qquad [e_1, e_3] = \alpha e_3 \qquad [e_2, e_3] = 0$$

for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Write $\mathcal{O}(\mathfrak{g}^*) = \mathbb{C}[x_1, x_2, x_3]$ following the notation of §2.6. The KKS Poisson structure on $\mathfrak{g}^*$ is given by the Poisson bracket on $\mathcal{O}(\mathfrak{g}^*)$ such that

$$\{x_1, x_2\} = x_2 \qquad \{x_1, x_3\} = \alpha x_3 \qquad \{x_2, x_3\} = 0.$$

As in [**53**, 1st. ed., Example II.2.37; 2nd. ed., Example II.2.43], the symplectic leaves in $\mathfrak{g}^*$ are the following sets:

- the individual points on the x_1-axis;
- the x_1x_2-plane minus the x_1-axis;
- the x_1x_3-plane minus the x_1-axis;
- the surfaces $(x_3 = \lambda x_2^\alpha \neq 0)$ for $\lambda \in \mathbb{C}^\times$.

Since α is irrational, the surfaces $(x_3 = \lambda x_2^\alpha \neq 0)$ are not algebraic – they are locally closed in the euclidean topology but not in the Zariski topology.

4. The Orbit Method from Lie theory

4.1. The Orbit Method. This term has been applied to a whole complex of methods in the representation theory of Lie groups and Lie algebras, and extended, as a guiding principle, to many other domains. To quote Kirillov's survey article [**30**],

> *The idea behind the orbit method is the unification of harmonic analysis with symplectic geometry (and it can also be considered as a part of the more general idea of the unification of mathematics and physics). In fact, this is a post factum formulation. Historically, the orbit method was proposed in* [**29**] *for the description of the unitary dual (i.e. the set of equivalence classes of unitary irreducible representations) of nilpotent Lie groups. It turned out that not only this problem but all other principal questions of representation theory—topological structure of the unitary dual, explicit description of the restriction and induction functors, character formulae, etc.—can be naturally answered in terms of coadjoint orbits.*

In Lie theory, the relevant orbits are defined as follows. Recall that if G is a Lie group with Lie algebra $\mathfrak{g}$, then G acts on $\mathfrak{g}$ by the *adjoint action* and on $\mathfrak{g}^*$ by the *coadjoint action*. The G-orbits of these actions are called *adjoint orbits* and *coadjoint orbits*, respectively. As a particular instance, the Orbit Method suggests that the primitive ideals of the enveloping algebra of $\mathfrak{g}$, being the kernels of the irreducible representations, should be related to the coadjoint orbits in $\mathfrak{g}^*$. Kirillov's original work provided the best such relationship – a bijection – when $\mathfrak{g}$ is nilpotent. There is also a bijection in case $\mathfrak{g}$ is solvable, except that the coadjoint orbits may have to be taken with respect to a different group than a Lie group with Lie algebra $\mathfrak{g}$. We discuss this situation in Section 5.

To place the coadjoint orbits in a geometric setting, view $\mathfrak{g}^*$ as the variety $\mathbb{A}^{\dim \mathfrak{g}}$, as in §2.6. We can then ask for a geometric description of these orbits within $\mathfrak{g}^*$. The answer is

a famous result discovered independently by Kirillov, Kostant, and Souriau (see, e.g., [**31**, §I.2.2, Theorem 2]):

4.2. Theorem. [Kirillov-Kostant-Souriau] *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then the coadjoint orbits of G in $\mathfrak{g}^*$ are precisely the symplectic leaves for the KKS Poisson structure.*

4.3. Example. Return to Example 3.8, and place the $x_1x_2x_3$-coordinates of points of $\mathfrak{g}^*$ in column vectors. We choose a Lie group G with Lie algebra $\mathfrak{g}$ as follows:

$$G := \left\{ \begin{bmatrix} 1 & u & v \\ 0 & t & 0 \\ 0 & 0 & t^\alpha \end{bmatrix} \;\middle|\; u, v \in \mathbb{C},\ t \in \mathbb{C}^\times \right\}.$$

The coadjoint action of G on $\mathfrak{g}^*$ can be identified with left multiplication of matrices from G on column vectors representing points in $\mathfrak{g}^*$. One easily checks that the G-orbits are exactly the symplectic leaves of $\mathfrak{g}^*$ identified in Example 3.8, as required by Theorem 4.2.

4.4. A general principle. In situations outside Lie theory, there may not be a suitable group action whose orbits play the role of coadjoint orbits. Instead, taking account of Theorem 4.2, one focusses on symplectic leaves. Restricting to the study of irreducible representations and primitive ideals, one is led to a general principle that we formulate as follows:

> *Given a noncommutative algebra A, relate the primitive ideals of A to the symplectic leaves corresponding to the Poisson structure on some associated algebraic variety arising from a semiclassical limit.*

This loose phrasing is intended to give the flavor of ideas coming out of the Orbit Method rather than to set up a precise recipe. Furthermore, this principle already requires modification in the case of enveloping algebras, and for general quantized coordinate rings.

On the other hand, the principle is right on target for the generic single parameter quantized coordinate rings $\mathcal{O}_q(G)$ of semisimple complex algebraic groups G, as conjectured by Hodges and Levasseur in [**22**, §2.8, Conjecture 1]: there is a bijection between the set of primitive ideals of $\mathcal{O}_q(G)$ and the set of symplectic leaves in G (for the semiclassical limit Poisson structure). They verified this conjecture for $G = SL_2(\mathbb{C})$ and $G = SL_3(\mathbb{C})$ in [**22**, Corollary B.2.2, Theorems 4.4.1, A.3.2], and then for $G = SL_n(\mathbb{C})$ in [**23**, Theorem 4.2 and following remarks]. For arbitrary connected, simply connected, semisimple complex Lie groups G, Joseph proved in [**27**, Theorem 9.2] that the primitive ideal space of $\mathcal{O}_q(G)$ has the form conjectured by Hodges and Levasseur in the first part of [**22**, §2.8, Conjecture 1], but he did not address connections with symplectic leaves. The full conjecture was established by Hodges, Levasseur, and Toro [**24**, Theorems 1.8, 4.18, Corollary 4.5] for connected semisimple complex Lie groups G. Their results also cover the multiparameter algebra $\mathcal{O}_{q,p}(G)$ under suitable algebraicity conditions on p.

4.5. Generic versus non-generic situations. As mentioned in the introduction, the principle discussed in §4.4 does not apply to non-generic quantized coordinate rings, which

typically have "too many" primitive ideals. The quantum plane provides the simplest illustration of this difficulty, and of the differences between the generic and non-generic cases. Take

$$A_q = \mathcal{O}_q(\mathbb{C}^2) = \mathbb{C}\langle x, y \mid xy = qyx \rangle,$$

where q is an arbitrary nonzero scalar in $\mathbb{C}$. By Example 2.2(a), the semiclassical limit of the family $(A_q)_{q \in \mathbb{C}^\times}$ is the polynomial ring $k[x, y]$, equipped with the Poisson bracket such that $\{x, y\} = xy$. It is easily checked that the corresponding symplectic leaves in $\mathbb{C}^2$ consist of

- the individual points on the x- and y-axes;
- the xy-plane minus the x- and y-axes.

If q is not a root of unity, one similarly checks that the primitive ideals of A_q consist of

- the maximal ideals $\langle x - \alpha,\, y \rangle$ and $\langle x,\, y - \beta \rangle$, for $\alpha, \beta \in \mathbb{C}$;
- the zero ideal.

(See [**4**, Example II.7.2], for instance, for details.) In this case, there is a natural bijection between the set of primitive ideals of A_q and the set of symplectic leaves in $\mathbb{C}^2$.

On the other hand, if q is a primitive l-th root of unity, the center of A_q equals the polynomial ring $\mathbb{C}[x^l, y^l]$, and A_q is a finitely generated $\mathbb{C}[x^l, y^l]$-module. In this case, the primitive ideals of A_q are maximal ideals, and they are parametrized (up to l-to-one) by the maximal ideals of $\mathbb{C}[x^l, y^l]$. While the set of primitive ideals of A_q has the same cardinality as the set of symplectic leaves in $\mathbb{C}^2$, there is no natural bijection, and certainly no homeomorphism if Zariski topologies are taken into account.

Such disparities occur in all the standard families of quantized coordinate rings, and provide just one of many distinctions between the generic and non-generic cases. We do not discuss the non-generic situation further, and concentrate on generic algebras.

5. Limitations of the Orbit Method for solvable Lie algebras

For a solvable finite dimensional complex Lie algebra $\mathfrak{g}$, the primitive ideals of the enveloping algebra $U(\mathfrak{g})$ are parametrized by means of the famous *Dixmier map*. At first glance, this is a successful instance of the Orbit Method, since the Dixmier map induces a bijection from a set of orbits in $\mathfrak{g}^*$ onto the set of primitive ideals of $U(\mathfrak{g})$. However, the relevant orbits are not, in general, those of the coadjoint action of a Lie group with Lie algebra $\mathfrak{g}$. Instead, the following group is needed.

5.1. The algebraic adjoint group. Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra. Treating $\mathfrak{g}$ for a moment just as a vector space, we have the general linear group $GL(\mathfrak{g})$ on $\mathfrak{g}$, which is a complex algebraic group whose Lie algebra is the general linear Lie algebra $\mathfrak{gl}(\mathfrak{g})$. Any algebraic subgroup of $GL(\mathfrak{g})$ (i.e., any Zariski closed subgroup) has a Lie algebra which is naturally contained in $\mathfrak{gl}(\mathfrak{g})$. The *algebraic adjoint group of* $\mathfrak{g}$ is the smallest algebraic subgroup $G \subseteq GL(\mathfrak{g})$ whose Lie algebra contains $\operatorname{ad}\mathfrak{g} = \{\operatorname{ad} x \mid x \in \mathfrak{g}\}$ (cf. [**2**, §12.2; **50**, Definition 24.8.1]).

The natural action of $GL(\mathfrak{g})$ on $\mathfrak{g}$ by linear automorphisms restricts to an action of G on $\mathfrak{g}$, the *adjoint action*. This, in turn, induces a (left) action of G on $\mathfrak{g}^*$, the *coadjoint*

action, under which

$$(g.\alpha)(x) = \alpha(g^{-1}.x)$$

for $g \in G$, $\alpha \in \mathfrak{g}^*$, and $x \in \mathfrak{g}$. The orbits of this action, the *coadjoint orbits*, are collected in the set $\mathfrak{g}^*/G$. We equip $\mathfrak{g}^*/G$ with the quotient topology induced from the Zariski topology on $\mathfrak{g}^*$, and thus refer to it as the *space* of coadjoint orbits.

5.2. Prime and primitive spectra. For any algebra A, we denote the collection of all primitive ideals of A by $\operatorname{prim} A$. This set supports a Zariski topology, under which the closed sets are the sets $V(I) := \{P \in \operatorname{prim} A \mid P \supseteq I\}$ for ideals I of A. We treat $\operatorname{prim} A$ as a topological space with this topology, and refer to it as the *primitive spectrum* of A. The analogous process, applied to the set of all prime ideals of A, results in the *prime spectrum* of A, denoted $\operatorname{spec} A$. Since primitive ideals are prime, $\operatorname{prim} A \subseteq \operatorname{spec} A$. In fact, $\operatorname{prim} A$ is a subspace of $\operatorname{spec} A$, that is, its topology coincides with the relative topology inherited from $\operatorname{spec} A$. Finally, we shall need the subspace of $\operatorname{spec} A$ consisting of all the maximal ideals of A. This is the *maximal ideal space* of A, denoted $\operatorname{maxspec} A$.

5.3. The Dixmier map. Let $\mathfrak{g}$ be a solvable finite dimensional complex Lie algebra. Following [**2**, §10.8], we use the name *Dixmier map* and the label Dx for the map

$$\mathrm{Dx} : \mathfrak{g}^* \longrightarrow \operatorname{prim} U(\mathfrak{g})$$

introduced by Dixmier in [**9**]. We do not give the definition here, but just refer to [**2**]. It turns out that this map is constant on G-orbits, and so it induces a *factorized Dixmier map*

$$\overline{\mathrm{Dx}} : \mathfrak{g}^*/G \longrightarrow \operatorname{prim} U(\mathfrak{g})$$

[**2**, §12.4]. Work of Dixmier, Conze, Duflo, and Rentschler led to the result that $\overline{\mathrm{Dx}}$ is a continuous bijection [**2**, Sätze 13.4, 15.1]. The conjecture that it is a homeomorphism was established later by Mathieu [**35**, Theorem], resulting in the following theorem:

5.4. Theorem. [Dixmier-Conze-Duflo-Rentschler-Mathieu] *Let $\mathfrak{g}$ be a solvable finite dimensional complex Lie algebra, and G its adjoint algebraic group. Then the factorized Dixmier map $\overline{\mathrm{Dx}}$ is a homeomorphism from $\mathfrak{g}^*/G$ onto $\operatorname{prim} U(\mathfrak{g})$.*

5.5. Algebraic versus non-algebraic cases. If $\mathfrak{g}$ is an *algebraic* Lie algebra, meaning that it is the Lie algebra of some algebraic group, then the adjoint algebraic group G is a Lie group, and its coadjoint orbits in $\mathfrak{g}^*$ are the symplectic leaves for the KKS Poisson structure, by Theorem 4.2. Otherwise, G is larger than the relevant Lie group, in the sense that its Lie algebra properly contains $\operatorname{ad}\mathfrak{g}$. In this case, its coadjoint orbits are larger too, typically larger than individual symplectic leaves. Our basic example illustrates this behavior.

5.6. Example. Return to Example 3.8, and again place the $x_1x_2x_3$-coordinates of points of $\mathfrak{g}^*$ in column vectors. The adjoint algebraic group G, written so as to act by left multiplication on column vectors, can be expressed as

$$G = \left\{ \begin{bmatrix} 1 & u & v \\ 0 & t & 0 \\ 0 & 0 & t' \end{bmatrix} \;\middle|\; u, v \in \mathbb{C},\ t, t' \in \mathbb{C}^\times \right\}$$

[**50**, §24.8.4]. The coadjoint orbits of G in $\mathfrak{g}^*$ are the following sets:

- the individual points on the x_1-axis;
- the x_1x_2-plane minus the x_1-axis;
- the x_1x_3-plane minus the x_1-axis;
- $\mathfrak{g}^*$ minus the x_1x_2- and x_1x_3-planes.

Comparing with Example 3.8, we see that the first three G-orbits are symplectic leaves, while the fourth is not. However, the fourth is at least a union of symplectic leaves.

The fourth G-orbit above is Zariski dense in $\mathfrak{g}^*$, while the others are not. Viewing these orbits as points in the orbit space $\mathfrak{g}^*/G$, we find that $\mathfrak{g}^*/G$ has a unique dense point (i.e., a unique dense singleton subset). By Theorem 5.4, the same holds for $\operatorname{prim} U(\mathfrak{g})$. (Translated into ideal theory, this means that there is one primitive ideal of $U(\mathfrak{g})$ which is contained in all other primitive ideals.) On the other hand, all the surfaces $(x_3 = \lambda x_2^\alpha \neq 0)$ are Zariski dense in $\mathfrak{g}^*$, and so the quotient topology on the space of symplectic leaves in $\mathfrak{g}^*$ has uncountably many dense points. Therefore this space of symplectic leaves cannot be homeomorphic to $\operatorname{prim} U(\mathfrak{g})$.

6. Poisson ideal theory and symplectic cores

Since the concept of symplectic leaves is differential-geometric, it should not be so surprising that it is not always suited to describe answers to algebraic problems, as seen in the previous section. Consequently, we look for an algebraic replacement. This is provided by Brown and Gordon's notion of *symplectic cores*, which is described via the ideal theory of Poisson algebras.

6.1. Poisson prime ideals. Let R be a (commutative) Poisson algebra (recall §3.1).

A *Poisson ideal* of R is any ideal I of the ring R which is also a Lie ideal relative to $\{-,-\}$, that is, $\{R, I\} \subseteq I$. Sums, products, and intersections of Poisson ideals are again Poisson ideals. Whenever I is a Poisson ideal of R, the Poisson bracket on R induces a well defined Poisson bracket on R/I, so that R/I becomes a Poisson algebra.

The *Poisson core* of an arbitrary ideal J of R is the largest Poisson ideal contained in J. This exists and is unique, because it is the sum of all Poisson ideals contained in J. We use $\mathcal{P}(J)$ to denote the Poisson core of J.

A *Poisson-prime ideal* of R is any proper Poisson ideal P of R with the following property: whenever the product of Poisson ideals I and J of R is contained in P, one of I or J must be contained in P. Obviously any prime Poisson ideal is Poisson-prime, but the converse can fail in positive characteristic. As we shall see in a moment, (Poisson-prime) is the same as (prime Poisson) when R is noetherian and k has characteristic zero; in that case, we will drop the hyphen and speak of *Poisson prime ideals*. Note also that if Q is an arbitrary prime ideal of R, then $\mathcal{P}(Q)$ is a Poisson-prime ideal.

The *Poisson-prime spectrum* of R, denoted $\operatorname{P.spec} R$, is the set of all Poisson-prime ideals of R, equipped with the natural Zariski-type topology, in which the closed sets are those of the form $V_P(I) := \{P \in \operatorname{P.spec} R \mid P \supseteq I\}$, for ideals I of R. It suffices to consider Poisson ideals in defining closed sets, since the ideal I in the definition of a closed set can

be replaced by the Poisson ideal it generates. (This observation is helpful in showing that finite unions of closed sets are closed.)

6.2. Lemma. *Let R be a Poisson k-algebra, where* $\operatorname{char} k = 0$. *Then the Poisson core of every prime ideal of R is prime, and all minimal prime ideals of R are Poisson ideals. If R is noetherian, the Poisson-prime ideals of R coincide with the prime Poisson ideals.*

Proof. Commutativity is not needed for this result. The commutative case is covered, for instance, by [**13**, Lemma 1.1], and the general case is proved the same way. We sketch the details for the reader's convenience.

The first conclusion is a consequence of [**10**, Lemma 3.3.2], and the second follows.

Now assume that R is noetherian, and let P be a Poisson-prime ideal of R. There exist prime ideals $Q_1, \dots, Q_t$ minimal over P such that $Q_1 Q_2 \cdots Q_t \subseteq P$. The minimal prime ideals Q_i/P in the Poisson algebra R/P must be Poisson ideals by what has been proved so far, and hence the Q_i are Poisson ideals of R. Poisson-primeness of P then implies that some $Q_j \subseteq P$, whence $P = Q_j$, proving that P is prime. □

6.3. Poisson-primitive ideals and symplectic cores. Let R be a (commutative) Poisson algebra.

The *Poisson-primitive ideals* of R are the Poisson cores of the maximal ideals of R. Note from §6.1 that all Poisson-primitive ideals are Poisson-prime.

This terminology is chosen to reflect the following parallel. An ideal P in an algebra A is left primitive if and only if P is the largest ideal contained in some maximal left ideal. If we view A as a (noncommutative) Poisson algebra via the commutator bracket $[-,-]$, then the ideals of A are precisely the Poisson left ideals. Thus, the left primitive ideals of A are exactly the Poisson cores of the maximal left ideals.

The *Poisson-primitive spectrum* of R, denoted $\operatorname{P.prim} R$, is the set of all Poisson-primitive ideals of R. This is a subset of $\operatorname{P.spec} R$, and we give it the relative topology.

By definition, the process of taking Poisson cores defines a surjective map

$$\operatorname{maxspec} R \longrightarrow \operatorname{P.prim} R,$$

and we note that this map is continuous. Its fibres, namely the sets

$$\{\mathfrak{m} \in \operatorname{maxspec} R \mid \mathcal{P}(\mathfrak{m}) = P\}$$

for $P \in \operatorname{P.prim} R$, are called *symplectic cores*. They determine a partition of $\operatorname{maxspec} R$.

Now suppose that $R = \mathcal{O}(V)$ is the coordinate ring of an affine variety V, and that k is algebraically closed. As in the complex case, we say that V is a *Poisson variety*. Since k is algebraically closed, there is a natural identification $V \equiv \operatorname{maxspec} R$, with which we transfer the symplectic cores from $\operatorname{maxspec} R$ to V. In other words, the *symplectic cores in V* are the sets

$$\{p \in V \mid \mathcal{P}(\mathfrak{m}_p) = P\}$$

for $P \in \operatorname{P.prim} R$, where $\mathfrak{m}_p = \{f \in R \mid f(p) = 0\}$.

6.4. Example. Return to Example 3.8, and set $R = \mathcal{O}(\mathfrak{g}^*) = \mathbb{C}[x_1, x_2, x_3]$. The Poisson-primitive ideals of R can be computed as follows:

$$\begin{aligned}
\mathcal{P}(\langle x_1 - \alpha, x_2, x_3 \rangle) &= \langle x_1 - \alpha, x_2, x_3 \rangle && (\alpha \in \mathbb{C}) \\
\mathcal{P}(\langle x_1 - \alpha, x_2 - \beta, x_3 \rangle) &= \langle x_3 \rangle && (\alpha \in \mathbb{C},\ \beta \in \mathbb{C}^\times) \\
\mathcal{P}(\langle x_1 - \alpha, x_2, x_3 - \gamma \rangle) &= \langle x_2 \rangle && (\alpha \in \mathbb{C},\ \gamma \in \mathbb{C}^\times) \\
\mathcal{P}(\langle x_1 - \alpha, x_2 - \beta, x_3 - \gamma \rangle) &= \langle 0 \rangle && (\alpha \in \mathbb{C},\ \beta, \gamma \in \mathbb{C}^\times).
\end{aligned}$$

It follows that the symplectic cores in $\mathfrak{g}^*$ are the sets

- the individual points on the x_1-axis;
- the x_1x_2-plane minus the x_1-axis;
- the x_1x_3-plane minus the x_1-axis;
- $\mathfrak{g}^*$ minus the x_1x_2- and x_1x_3-planes.

These are precisely the coadjoint orbits of the adjoint algebraic group of $\mathfrak{g}$, as we saw in Example 5.6.

7. Symplectic cores versus symplectic leaves

Symplectic cores are related to symplectic leaves by the following result of Brown and Gordon [**6**, Lemma 3.3 and Proposition 3.6]; further relations will be given below. Here "locally closed" refers to the Zariski topology.

7.1. Theorem. [Brown-Gordon] *Let V be a complex affine Poisson variety.*

(a) *Each symplectic core in V is locally closed, and is a union of symplectic leaves.*

(b) *If the symplectic leaves in V are all locally closed, then they coincide with the symplectic cores.*

It is a standard result that the orbits of a connected algebraic group G acting on a variety X can be recovered from the orbit closures, as follows. Take any orbit closure C, and remove all orbit closures properly contained in C. The result will be a single G-orbit, and all G-orbits in X are obtained by this means. Yakimov has conjectured that the symplectic cores in a complex affine Poisson variety can be recovered from the closures of the symplectic leaves in a similar manner. We verify this below, with the help of the following lemma of Brown and Gordon [**6**, Lemma 3.5]. All topological properties are to be taken relative to the Zariski topology.

7.2. Lemma. [Brown-Gordon] *Let V be a complex affine Poisson variety, and $R = \mathcal{O}(V)$. Let L be a symplectic leaf in V, and set $K = \{f \in R \mid f = 0 \text{ on } L\}$. Then K is a Poisson-primitive ideal of R, and L is contained in the corresponding symplectic core, that is, $\mathcal{P}(\mathfrak{m}_p) = K$ for all $p \in L$.*

7.3. Lemma. *Let V be a complex affine Poisson variety, and $R = \mathcal{O}(V)$. Let K be a Poisson ideal of R, and X the closed subvariety of V determined by K. Then X is a union of symplectic cores and a union of symplectic leaves. In particular, the closure of any symplectic leaf of V is a union of symplectic leaves.*

Proof. If $p \in X$, then $\mathfrak{m}_p \supseteq K$. Since K is a Poisson ideal, it must be contained in the Poisson-primitive ideal $P = \mathcal{P}(\mathfrak{m}_p)$. Now the set $C = \{q \in V \mid \mathcal{P}(\mathfrak{m}_q) = P\}$ is the symplectic core containing p, and $C \subseteq X$ because $\mathfrak{m}_q \supseteq P \supseteq K$ for all $q \in C$. Therefore X is a union of symplectic cores. That X is a union of symplectic leaves now follows from Theorem 7.1(a).

For any symplectic leaf L of V, the ideal I of functions in R that vanish on L is a Poisson ideal by Lemma 7.2. The closed subvariety determined by I is the closure of L, and this is a union of symplectic leaves by what we have just proved. □

We can now prove that symplectic cores are obtained from symplectic leaves in the manner proposed by Yakimov; this is parts (c) and (e) of the following theorem. Here overbars denote closures.

7.4. Theorem. *Let V be a complex affine Poisson variety, and L a symplectic leaf in V.*

(a) *There is a unique symplectic core C in V containing L, and $C \subseteq \overline{L}$.*

(b) *C is the union of those symplectic leaves of V which are dense in $\overline{L}$.*

(c) *$C = \overline{L} \setminus \bigcup_M \overline{M}$ where M runs over those symplectic leaves whose closures are properly contained in $\overline{L}$.*

(d) *C is the unique symplectic core dense in $\overline{L}$.*

(e) *Each symplectic core in V is dense in the closure of every symplectic leaf it contains. Hence, it can be obtained from the closure of such a leaf as in part* (c).

Proof. Set $R = \mathcal{O}(V)$, and let K be the ideal of functions in R that vanish on L.

(a) The symplectic cores and the symplectic leaves both partition V, and the latter form a finer partition, by Theorem 7.1(a). This implies the existence and uniqueness of C.

By Lemma 7.2, K is a Poisson-primitive ideal, and the symplectic core it determines contains L. By uniqueness, this core is C, that is, $C = \{p \in V \mid \mathcal{P}(\mathfrak{m}_p) = K\}$. In particular, $\mathfrak{m}_p \supseteq K$ for all $p \in C$, from which it follows that $C \subseteq \overline{L}$.

(b) If M is a symplectic leaf which is dense in $\overline{L}$, then K equals the ideal of functions in R that vanish on M, and Lemma 7.2 implies that $M \subseteq C$. On the other hand, if M' is a symplectic leaf which is contained in but not dense in $\overline{L}$, the ideal K' of functions vanishing on M' properly contains K, whence M' is contained in a symplectic core different from C. In this case, M' is disjoint from C. Part (b) now follows, because $\overline{L}$ is a union of symplectic leaves, by Lemma 7.3.

(c) In view of Lemma 7.3, the given union $\bigcup_M \overline{M}$ equals the union of those symplectic leaves which are contained in $\overline{L}$ but not dense in $\overline{L}$. The given formula for C thus follows from part (b).

(d) Clearly C is dense in $\overline{L}$, since $L \subseteq C \subseteq \overline{L}$. If D is a different symplectic core contained in $\overline{L}$, then by (b), any symplectic leaf $N \subseteq D$ is not dense in $\overline{L}$. But $D \subseteq \overline{N}$ by (a), and thus D is not dense in $\overline{L}$.

(e) Suppose that D is a symplectic core in V, and N a symplectic leaf contained in D. By (a), D is the unique symplectic core containing N, and $D \subseteq \overline{N}$, whence D is dense in $\overline{N}$. The final statement now follows from (c), with C and L replaced by D and N. □

8. Symplectic cores versus primitive ideals for solvable Lie algebras

We now show that the concept of symplectic cores exactly overcomes the limitations of symplectic leaves with respect to the Dixmier map discussed in Section 5. Namely, the Dixmier map provides a homeomorphism from the space of symplectic cores in $\mathfrak{g}^*$ onto the primitive spectrum of $U(\mathfrak{g})$, for any solvable finite dimensional complex Lie algebra $\mathfrak{g}$. This just amounts to showing that the coadjoint orbits in $\mathfrak{g}^*$, with respect to the adjoint algebraic group of $\mathfrak{g}$, coincide with the symplectic cores. Solvability is not needed for the latter result.

All that is required to obtain the new statement about the Dixmier map is to reinterpret parts of the development of Theorem 5.4 in terms of the new concepts. This reinterpretation also shows that (for $\mathfrak{g}$ solvable) $\mathrm{P.prim}\,\mathcal{O}(\mathfrak{g}^*)$ is homeomorphic to $\mathrm{prim}\,U(\mathfrak{g})$. With a little extra effort, we can handle prime ideals as well, showing that $\mathrm{P.spec}\,\mathcal{O}(\mathfrak{g}^*)$ is homeomorphic to $\mathrm{spec}\,U(\mathfrak{g})$.

Throughout this section, $\mathfrak{g}$ will denote a finite dimensional complex Lie algebra and G its adjoint algebraic group. We do not assume $\mathfrak{g}$ solvable until Theorem 8.5. Some of the results we will need are developed in the literature in terms of $S(\mathfrak{g})$ rather than $\mathcal{O}(\mathfrak{g}^*)$. This requires use of the Poisson isomorphism $\theta : S(\mathfrak{g}) \xrightarrow{\cong} \mathcal{O}(\mathfrak{g}^*)$ of (2.6).

8.1. Actions of G and $\mathfrak{g}$. The group G acts on $\mathfrak{g}$ and $\mathfrak{g}^*$ by the adjoint and coadjoint actions, respectively, as in §5.1. In turn, these induce actions of G by $\mathbb{C}$-algebra automorphisms on $S(\mathfrak{g})$ and $\mathcal{O}(\mathfrak{g}^*)$, actions which we also refer to as *adjoint* and *coadjoint actions*. All G-actions we mention will refer to one of these four cases. Let us write $\mathrm{spec}^{\mathrm{G}}\,S(\mathfrak{g})$ and $\mathrm{spec}^{\mathrm{G}}\,\mathcal{O}(\mathfrak{g}^*)$ for the sets of G-stable prime ideals in $S(\mathfrak{g})$ and $\mathcal{O}(\mathfrak{g}^*)$, respectively, equipped with the relative topologies from $\mathrm{spec}\,S(\mathfrak{g})$ and $\mathrm{spec}\,\mathcal{O}(\mathfrak{g}^*)$.

We claim that the isomorphism θ is G-equivariant. To see this, let $\{e_1, \dots, e_n\}$ be a basis for $\mathfrak{g}$ and $\{\alpha_1, \dots, \alpha_n\}$ the corresponding dual basis for $\mathfrak{g}^*$. As in §2.6, $\mathcal{O}(\mathfrak{g}^*) = \mathbb{C}[x_1, \dots, x_n]$ where each $x_i = \theta(e_i)$. Given $\gamma \in G$, there are scalars $\gamma_{ij} \in \mathbb{C}$ such that $\gamma.e_j = \sum_i \gamma_{ij} e_i$ for all j. Consequently,

$$(\gamma.x_j)(\alpha_i) = x_j(\gamma^{-1}.\alpha_i) = (\gamma^{-1}.\alpha_i)(e_j) = \alpha_i(\gamma.e_j) = \gamma_{ij}$$

for all i, j, from which we conclude that $\gamma.x_j = \sum_i \gamma_{ij} x_i$ for all j. Therefore $\gamma.\theta(e_j) = \theta(\gamma.e_j)$ for all j, and the G-equivariance of θ follows.

For each $e \in \mathfrak{g}$, the Lie derivation $\mathrm{ad}\, e = [e, -]$ on $\mathfrak{g}$ extends uniquely to a derivation on $S(\mathfrak{g})$, namely the Hamiltonian $\{e, -\}$. This yields an action of $\mathfrak{g}$ on $S(\mathfrak{g})$ by derivations. We write $\mathrm{spec}^{\mathfrak{g}}\,S(\mathfrak{g})$ for the set of $\mathfrak{g}$-stable prime ideals of $S(\mathfrak{g})$, equipped with the relative topology from $\mathrm{spec}\,S(\mathfrak{g})$.

8.2. Lemma. (a) $\mathrm{spec}^{\mathrm{G}}\,S(\mathfrak{g}) = \mathrm{spec}^{\mathfrak{g}}\,S(\mathfrak{g}) = \mathrm{P.spec}\,S(\mathfrak{g})$.

(b) $\mathrm{spec}^{\mathrm{G}}\,\mathcal{O}(\mathfrak{g}^*) = \mathrm{P.spec}\,\mathcal{O}(\mathfrak{g}^*)$.

(c) *θ induces a homeomorphism* $\mathrm{spec}^{\mathfrak{g}}\,S(\mathfrak{g}) \xrightarrow{\approx} \mathrm{P.spec}\,\mathcal{O}(\mathfrak{g}^*)$.

Proof. (a) Since $\mathfrak{g}$ generates the algebra $S(\mathfrak{g})$, the $\mathfrak{g}$-stable ideals of $S(\mathfrak{g})$ coincide with the Poisson ideals. Hence, $\mathrm{spec}^{\mathfrak{g}}\,S(\mathfrak{g}) = \mathrm{P.spec}\,S(\mathfrak{g})$. By [**2**, §13.1] or [**50**, §24.8.3], the

$\mathfrak{g}$-stable ideals of $S(\mathfrak{g})$ coincide with the G-stable ideals. From this, we immediately obtain $\operatorname{spec}^{G} S(\mathfrak{g}) = \operatorname{spec}^{\mathfrak{g}} S(\mathfrak{g})$.

(b)(c) These follow immediately from (a), because θ is both G-equivariant and a Poisson isomorphism. $\square$

Following our previous notation for maximal ideals corresponding to points in varieties, write $\mathfrak{m}_\alpha$ for the maximal ideal of $\mathcal{O}(\mathfrak{g}^*)$ corresponding to a point $\alpha \in \mathfrak{g}^*$.

8.3. Proposition. *Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra and G its adjoint algebraic group. There is a homeomorphism $\phi : \mathfrak{g}^*/G \to \operatorname{P.prim} \mathcal{O}(\mathfrak{g}^*)$ such that $\phi(G.\alpha) = \mathcal{P}(\mathfrak{m}_\alpha)$ for all $\alpha \in \mathfrak{g}^*$.*

Proof. Since $S(\mathfrak{g})$ is isomorphic to $\mathcal{O}(\mathfrak{g}^*)$, its maximal ideal space is homeomorphic to $\mathfrak{g}^*$. A coordinate-free way to express the inverse isomorphism is to send each $\alpha \in \mathfrak{g}^*$ to the ideal $\underline{m}_\alpha = \langle e - \alpha(e) \mid e \in \mathfrak{g} \rangle$ of $S(\mathfrak{g})$. Observe that $\theta(\underline{m}_\alpha) = \mathfrak{m}_\alpha$.

By [**2**, Lemma 13.2 and proof], there is a topological embedding

$$\tau : \mathfrak{g}^*/G \longrightarrow \operatorname{spec}^{\mathfrak{g}} S(\mathfrak{g})$$

such that $\tau(G.\alpha) = \bigcap_{\gamma \in G} \gamma.\underline{m}_\alpha$ for $\alpha \in \mathfrak{g}^*$. Thus, $\tau(G.\alpha)$ is the largest G-stable ideal of $S(\mathfrak{g})$ contained in $\underline{m}_\alpha$. Invoking [**2**, §13.1] or [**50**, §24.8.3] again, we find that $\tau(G.\alpha)$ is the largest $\mathfrak{g}$-stable ideal of $S(\mathfrak{g})$ contained in $\underline{m}_\alpha$. In particular, it now follows from [**10**, Lemma 3.3.2] that $\tau(G.\alpha)$ is a prime ideal. Hence, we can say that $\tau(G.\alpha)$ equals the largest member of $\operatorname{spec}^{G} S(\mathfrak{g})$ contained in $\underline{m}_\alpha$. Since θ is G-equivariant, it follows that $\theta\tau(G.\alpha)$ equals the largest member of $\operatorname{spec}^{G} \mathcal{O}(\mathfrak{g}^*)$ contained in $\mathfrak{m}_\alpha$. In view of Lemma 8.2(b), we conclude that $\theta\tau(G.\alpha) = \mathcal{P}(\mathfrak{m}_\alpha)$.

Combining the above with Lemma 8.2(c), we obtain a topological embedding

$$\phi : \mathfrak{g}^*/G \to \operatorname{P.spec} \mathcal{O}(\mathfrak{g}^*)$$

such that $\phi(G.\alpha) = \mathcal{P}(\mathfrak{m}_\alpha)$ for $\alpha \in \mathfrak{g}^*$. Since the image of ϕ is, by definition, $\operatorname{P.prim} \mathcal{O}(\mathfrak{g}^*)$, the proposition is proved. $\square$

8.4. Corollary. *Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra and G its adjoint algebraic group. The G-orbits in $\mathfrak{g}^*$ are precisely the symplectic cores.*

Proof. Injectivity and well-definedness of the homeomorphism ϕ of Proposition 8.3 say that for all $\alpha, \beta \in \mathfrak{g}^*$, we have $G.\alpha = G.\beta$ if and only if $\mathcal{P}(\mathfrak{m}_\alpha) = \mathcal{P}(\mathfrak{m}_\beta)$. Thus, α and β lie in the same G-orbit if and only if they lie in the same symplectic core. $\square$

Corollary 8.4 allows us to phrase the Dixmier-Conze-Duflo-Rentschler-Mathieu Theorem in terms of symplectic cores:

8.5. Theorem. *Let $\mathfrak{g}$ be a solvable finite dimensional complex Lie algebra, and let X be the set of symplectic cores in $\mathfrak{g}^*$, with the quotient topology induced from $\mathfrak{g}^*$. Then the Dixmier map induces a homeomorphism $X \to \operatorname{prim} U(\mathfrak{g})$.* $\square$

8.6. The extended Dixmier map. Continue to assume that $\mathfrak{g}$ is solvable. Via the embedding $\mathfrak{g}^*/G \longrightarrow \operatorname{spec}^{\mathfrak{g}} S(\mathfrak{g})$ from [**2**, Lemma 13.2] used above, identify $\mathfrak{g}^*/G$ with a subspace of $\operatorname{spec}^{\mathfrak{g}} S(\mathfrak{g})$. Borho, Gabriel, and Rentschler showed that $\overline{\mathrm{Dx}}$ extends uniquely to a continuous map

$$\widetilde{\mathrm{Dx}} : \operatorname{spec}^{\mathfrak{g}} S(\mathfrak{g}) \to \operatorname{spec} U(\mathfrak{g}),$$

given by the rule

$$\widetilde{\mathrm{Dx}}(P) = \bigcap \{\mathrm{Dx}(\alpha) \mid \alpha \in \mathfrak{g}^* \text{ and } \underline{m}_\alpha \supseteq P\}$$

for $P \in \operatorname{spec}^{\mathfrak{g}} S(\mathfrak{g})$ [**2**, Satz 13.4]. They named this the *extended Dixmier map*, and proved that it is a continuous bijection [**2**, Satz 13.4, Kor. 15.1]. Their methods, combined with Mathieu's theorem, imply that $\widetilde{\mathrm{Dx}}$ is a homeomorphism, as we will see shortly.

8.7. Quasi-homeomorphisms and sauber spaces. Let X and Y be topological spaces. A continuous map $\phi : X \to Y$ is a *quasi-homeomorphism* provided the induced map $F \mapsto \phi^{-1}(F)$ is an isomorphism from the lattice of closed subsets of Y onto the lattice of closed subsets of X. If X is a subspace of Y, the inclusion map $X \to Y$ is a quasi-homeomorphism if and only if $\overline{F \cap X} = F$ for all closed sets $F \subseteq Y$ [**2**, §1.6]. Borho, Gabriel, and Rentschler observed that the inclusion map $\operatorname{prim} U(\mathfrak{g}) \to \operatorname{spec} U(\mathfrak{g})$ is a quasi-homeomorphism [**2**, Beispiel 1.6], as is the above embedding $\mathfrak{g}^*/G \longrightarrow \operatorname{spec}^{\mathfrak{g}} S(\mathfrak{g})$ [**2**, Lemma 13.2].

A *generic point* of a closed subset $F \subseteq X$ is any point $x \in F$ such that $F = \overline{\{x\}}$. The space X is *sauber* (English: *tidy*) provided every irreducible closed subset of X has precisely one generic point. As observed in [**2**, §13.3], the prime spectrum of any noetherian ring is sauber. We include the short argument in the lemma below, for the reader's convenience. The same argument shows that $\operatorname{spec}^{\mathfrak{g}} S(\mathfrak{g})$ is sauber. These spaces are noetherian as well, since they have Zariski topologies arising from noetherian rings.

8.8. Lemma. *Let A be a noetherian ring and R a commutative noetherian Poisson k-algebra, with* $\operatorname{char} k = 0$.

(a) *The prime spectrum* $\operatorname{spec} A$ *is a sauber noetherian space, and if A is a Jacobson ring, the inclusion map* $\operatorname{prim} A \to \operatorname{spec} A$ *is a quasi-homeomorphism.*

(b) *The Poisson prime spectrum* $\operatorname{P.spec} R$ *is a sauber noetherian space, and if R is an affine k-algebra, the inclusion map* $\operatorname{P.prim} R \to \operatorname{P.spec} R$ *is a quasi-homeomorphism.*

Proof. (a) Suppose that $F_1 \supseteq F_2 \supseteq \cdots$ is a decreasing sequence of closed sets in $\operatorname{spec} A$. We may write each $F_j = V(I_j)$ where $I_j = \bigcap F_j$. Then $I_1 \subseteq I_2 \subseteq \cdots$ is an increasing sequence of ideals of A. Since this sequence stabilizes, so does the original sequence of closed sets. Thus, $\operatorname{spec} A$ is a noetherian space.

Let $F = V(I)$ be an arbitrary closed subset of $\operatorname{spec} A$, where I is an ideal of A. We may replace I by its prime radical, so there is no loss of generality in assuming that I is semiprime. Since A is noetherian, there are only finitely many prime ideals minimal over I, say $Q_1, \dots, Q_n$, and $I = Q_1 \cap \cdots \cap Q_n$. It follows that $F = V(Q_1) \cup \cdots \cup V(Q_n)$.

If F is irreducible, then $F = V(Q_j)$ for some j. In this case, Q_j is the unique generic point of F, proving that $\operatorname{spec} A$ is sauber.

Now assume that A is a Jacobson ring, so that all prime ideals of A are intersections of primitive ideals. It follows that

$$I = \bigcap F = \bigcap (F \cap \operatorname{prim} A),$$

from which we see that F equals the closure of $F \cap \operatorname{prim} A$ in $\operatorname{spec} A$. Thus, by [**2**, §1.6], the inclusion map $\operatorname{prim} A \to \operatorname{spec} A$ is a quasi-homeomorphism.

(b) The argument applied in (a) also shows that $\operatorname{P.spec} R$ is a noetherian space.

As discussed in §6.1, any closed set F in $\operatorname{P.spec} R$ can be written $F = V_P(I)$ for some Poisson ideal I. There are only finitely many prime ideals minimal over I, say $Q_1, \dots, Q_n$, and the Q_i are Poisson ideals by Lemma 6.2. Hence, we may replace I by $Q_1 \cap \cdots \cap Q_n$, and it follows that $F = V_P(Q_1) \cup \cdots \cup V_P(Q_n)$.

Just as in (a), if F is irreducible, $F = V_P(Q_j)$ for some j, and then Q_j is the unique generic point of F. This proves that $\operatorname{P.spec} R$ is sauber.

Now assume that R is an affine k-algebra. Then R is a Jacobson ring, and it follows that every Poisson prime ideal of R is an intersection of Poisson-primitive ideals (e.g., see [**13**, Lemma 1.1(e)]). From this, we conclude as in (a) that the inclusion map $\operatorname{P.prim} R \to \operatorname{P.spec} R$ is a quasi-homeomorphism. □

8.9. Lemma. *Let $X \subseteq X'$ and $Y \subseteq Y'$ be topological spaces, such that X' and Y' are sauber and noetherian. Assume also that the inclusion maps $X \to X'$ and $Y \to Y'$ are quasi-homeomorphisms. Then any continuous map $\phi : X \to Y$ extends uniquely to a continuous map $\phi' : X' \to Y'$. Moreover, if ϕ is a homeomorphism, so is ϕ'.*

Proof. The existence and uniqueness of ϕ are proved in [**2**, Lemma 13.3]. The final statement follows by the usual universal property argument. □

8.10. Theorem. [Borho-Gabriel-Rentschler-Mathieu] *Let $\mathfrak{g}$ be a solvable finite dimensional complex Lie algebra. The extended Dixmier map*

$$\widetilde{\mathrm{Dx}} : \operatorname{spec}^{\mathfrak{g}} S(\mathfrak{g}) \longrightarrow \operatorname{spec} U(\mathfrak{g})$$

is a homeomorphism.

Proof. Following the proof of [**2**, Satz 13.4], recall that $\operatorname{spec}^{\mathfrak{g}} S(\mathfrak{g})$ and $\operatorname{spec} U(\mathfrak{g})$ are sauber noetherian spaces, and that the embedding $\mathfrak{g}^*/G \to \operatorname{spec}^{\mathfrak{g}} S(\mathfrak{g})$ and the inclusion $\operatorname{prim} U(\mathfrak{g}) \to \operatorname{spec} U(\mathfrak{g})$ are quasi-homeomorphisms. The map $\widetilde{\mathrm{Dx}}$ is defined, with the help of Lemma 8.9, to be the unique continuous map from $\operatorname{spec}^{\mathfrak{g}} S(\mathfrak{g})$ to $\operatorname{spec} U(\mathfrak{g})$ extending $\overline{\mathrm{Dx}}$. Since $\overline{\mathrm{Dx}}$ is a homeomorphism, Lemma 8.9 implies that $\widetilde{\mathrm{Dx}}$ is a homeomorphism. □

In Poisson-ideal-theoretic terms, Theorems 5.4 and 8.10 can be restated as follows.

8.11. Theorem. *Let $\mathfrak{g}$ be a solvable finite dimensional complex Lie algebra. Then there is a homeomorphism*

$$\psi : \operatorname{P.prim} \mathcal{O}(\mathfrak{g}^*) \longrightarrow \operatorname{prim} U(\mathfrak{g})$$

such that $\psi(\mathcal{P}(\mathfrak{m}_\alpha)) = \mathrm{Dx}(\alpha)$ *for* $\alpha \in \mathfrak{g}^*$*, and* ψ *extends uniquely to a homeomorphism*

$$\mathrm{P.spec}\,\mathcal{O}(\mathfrak{g}^*) \longrightarrow \mathrm{spec}\,U(\mathfrak{g}).$$

Proof. To obtain ψ, just compose the factorized Dixmier map $\overline{\mathrm{Dx}}$ with the inverse of the homeomorphism ϕ of Proposition 8.3. By Lemma 8.8, $\mathrm{P.spec}\,\mathcal{O}(\mathfrak{g}^*)$ and $\mathrm{spec}\,U(\mathfrak{g})$ are sauber noetherian spaces, and the inclusion maps $\mathrm{P.prim}\,\mathcal{O}(\mathfrak{g}^*) \to \mathrm{P.spec}\,\mathcal{O}(\mathfrak{g}^*)$ and $\mathrm{prim}\,U(\mathfrak{g}) \to \mathrm{spec}\,U(\mathfrak{g})$ are quasi-homeomorphisms. Therefore the existence and uniqueness of the desired extension of ψ follow from Lemma 8.9. $\square$

9. Modified conjectures for quantized coordinate rings

In light of Theorems 7.1, 7.4, 8.5, and 8.11, we nominate the concept of symplectic cores as the *best algebraic approximation* for symplectic leaves. Further, we suggest that symplectic leaves should be replaced by symplectic cores in applications of the Orbit Method to algebraic problems. In particular, we revise and refine the general principle discussed in §4.4 to the following conjecture. It is, of necessity, somewhat imprecise, given the lack of a precise definition of the concept of quantized coordinate rings.

9.1. Primitive spectrum conjecture for quantized coordinate rings.

> *Assume that* k *is algebraically closed of characteristic zero, and let* A *be a generic quantized coordinate ring of an affine algebraic variety* V *over* k*. Then* A *should be a member of a flat family of* k*-algebras with semiclassical limit* $\mathcal{O}(V)$*, such that* $\mathrm{prim}\,A$ *is homeomorphic to the space of symplectic cores in* V*, with respect to the semiclassical limit Poisson structure. Further, there should be compatible homeomorphisms* $\mathrm{prim}\,A \to \mathrm{P.prim}\,\mathcal{O}(V)$ *and* $\mathrm{spec}\,A \to \mathrm{P.spec}\,\mathcal{O}(V)$.

Each of the known types of quantized coordinate rings supports an action of an algebraic torus $H = (k^\times)^m$ (see [**4**, §§II.1.14-18] for a summary), which has a parallel action (by Poisson automorphisms) on the semiclassical limit (e.g., see [**19**, §0.2; **14**, Section 2]). We tighten the conjecture above and posit that there should exist homeomorphisms as desribed which are also equivariant with respect to the relevant torus actions.

9.2. Remarks. (a) The discussion of the simple example $A_q = \mathcal{O}_q(\mathbb{C}^2)$ in §4.5 indicates why Conjecture 9.1 is restricted to generic quantized coordinate rings. In particular, $\mathrm{prim}\,A_q$ has a generic point when q is not a root of unity, but no generic points otherwise. Since $\mathrm{P.spec}\,\mathbb{C}[x,y]$ has a generic point, it is not homeomorphic to $\mathrm{prim}\,A_q$ when q is a root of unity.

(b) Each of the "standard" single parameter quantized coordinate rings is defined as a member of a one-parameter family of algebras, and it is this (flat) family to which the conjecture is meant to apply. For instance, the algebras $\mathcal{O}_q(SL_n(k))$ (with n fixed) are defined for all $q \in k^\times$ in the same way (e.g., [**4**, §I.2.4]), and substituting an indeterminate t for q in the definition results in a torsionfree $k[t^{\pm 1}]$-algebra A with $A/(t-q)A \cong \mathcal{O}_q(SL_n(k))$ for all $q \in k^\times$, just as with the case $n = 2$ in §§1.6, 2.2(c). The semiclassical limit is

$\mathcal{O}(SL_n(k))$ with the Poisson bracket satisfying

$$\begin{aligned}
\{X_{ij}, X_{im}\} &= X_{ij}X_{im} && (j < m)\\
\{X_{ij}, X_{lj}\} &= X_{ij}X_{lj} && (i < l)\\
\{X_{ij}, X_{lm}\} &= 0 && (i < l,\ j > m)\\
\{X_{ij}, X_{lm}\} &= 2X_{im}X_{lj} && (i < l,\ j < m).
\end{aligned}$$

This Poisson structure and the above flat family should feature in the SL_n case of Conjecture 9.1, that is, for q not a root of unity, $\operatorname{prim}\mathcal{O}_q(SL_n(k))$ should be homeomorphic to the space of symplectic cores in $SL_n(k)$ and to $\text{P.prim}\,\mathcal{O}(SL_n(k))$, and $\operatorname{spec}\mathcal{O}_q(SL_n(k))$ should be homeomorphic to $\text{P.spec}\,\mathcal{O}(SL_n(k))$. Such a "standard" version of the conjecture is to be posed for $\mathcal{O}_q(M_n(k))$, $\mathcal{O}_q(GL_n(k))$, $\mathcal{O}_q(G)$, and other "standard" cases.

The situation is more involved for "nonstandard" cases, and for multiparameter families, which have to be reduced to single parameter families in order to obtain semiclassical limits. In such cases, the conjecture may be sensitive to the choice of flat family – different flat families may yield different Poisson structures in the semiclassical limit, and the conjecture may hold for some of these semiclassical limits but not for others. This phenomenon appears in an example of Vancliff [**52**, Example 3.14], which we discuss in Example 9.9.

(c) As discussed at the end of Example 2.6, the enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$ is a generic member of the flat family given by the homogenization of $U(\mathfrak{g})$, and so $U(\mathfrak{g})$ should qualify as a generic quantized coordinate ring of $\mathfrak{g}^*$. The semiclassical limit of this family is the Poisson algebra $\mathcal{O}(\mathfrak{g}^*)$. For this setting, K. A. Brown has noted difficulties with Conjecture 9.1 in what one might expect to be the most canonical case, namely when $\mathfrak{g}$ is semisimple [**3**]. Following the Orbit Method, one would seek a bijection $\mathcal{L} \longleftrightarrow P$ between symplectic leaves in $\mathfrak{g}^*$ and primitive ideals in $U(\mathfrak{g})$ such that the Gelfand-Kirillov dimension of $U(\mathfrak{g})/P$ equals the dimension of $\mathcal{L}$. In particular, the zero-dimensional symplectic leaves of $\mathfrak{g}^*$, which are the same as the zero-dimensional symplectic cores, should match up with the maximal ideals of finite codimension in $U(\mathfrak{g})$. However, $U(\mathfrak{g})$ has infinitely many such maximal ideals, while there is only one zero-dimensional symplectic leaf in $\mathfrak{g}^*$. (The latter can be verified by using Theorem 4.2 together with the fact that the identification of $\mathfrak{g}^*$ with $\mathfrak{g}$ via the Killing form identifies the coadjoint orbits in $\mathfrak{g}^*$ with the adjoint orbits in $\mathfrak{g}$ [**8**, p. 12].)

Other differences are already visible in the case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. As is easily computed, all but one of the coadjoint orbits in $\mathfrak{g}^*$ are closed (compare with the adjoint orbits, computed in [**8**, Example 1.2.1]). It follows (using Proposition 8.3, or by direct computation) that all but one of the points of $\text{P.prim}\,\mathcal{O}(\mathfrak{g}^*)$ are closed. However, $\operatorname{prim} U(\mathfrak{g})$ has infinitely many non-closed points, and therefore it is not homeomorphic to $\text{P.prim}\,\mathcal{O}(\mathfrak{g}^*)$.

(d) Whenever Conjecture 9.1 does hold, the space of symplectic cores in V must be homeomorphic to $\text{P.prim}\,\mathcal{O}(V)$. It is an open question whether the space of symplectic cores for an arbitrary affine Poisson algebra R is homeomorphic to $\text{P.prim}\,R$, but this does hold when R satisfies the *Poisson Dixmier-Moeglin equivalence*, as follows from [**13**, Theorem 1.5]; we excerpt the basic argument in Lemma 9.3. This equivalence requires that the Poisson-primitive ideals of R coincide with the locally closed points of $\text{P.spec}\,R$,

and with those Poisson prime ideals P of R for which the Poisson center (cf. §9.6(b)) of the quotient field of R/P is algebraic over k. It holds for the semiclassical limits of many quantized coordinate rings via [**13**, Theorem 4.1], as shown in [**14**, Section 2].

(e) As in Theorem 8.11, the existence of a homeomorphism $\operatorname{prim} A \to \operatorname{P.prim} \mathcal{O}(V)$ as in the conjecture typically implies the existence of a compatible homeomorphism $\operatorname{spec} A \to \operatorname{P.spec} \mathcal{O}(V)$. We display this in Lemma 9.4 below for emphasis. On the other hand, a homeomorphism $\operatorname{spec} A \to \operatorname{P.spec} \mathcal{O}(V)$ will restrict to a homeomorphism $\operatorname{prim} A \to \operatorname{P.prim} \mathcal{O}(V)$ provided $\mathcal{O}(V)$ satisfies the Poisson Dixmier-Moeglin equivalence and A satisfies the *Dixmier-Moeglin equivalence*. The latter equivalence requires that the primitive ideals of A coincide with the locally closed points of $\operatorname{spec} A$, and with those prime ideals P of A for which $Z(\operatorname{Fract} A/P)$ is algebraic over k. It was verified for many quantized coordinate rings in [**16**] (see [**4**, Corollary II.8.5] for a summary).

9.3. Lemma. *Let R be a commutative affine Poisson k-algebra, and assume that all Poisson-primitive ideals of R are locally closed points in* $\operatorname{P.spec} R$. *Then the Zariski topology on* $\operatorname{P.prim} R$ *coincides with the quotient topology induced by the Poisson core map* $\mathcal{P}(-) : \operatorname{maxspec} R \to \operatorname{P.prim} R$. *Consequently, the space of symplectic cores in* $\operatorname{maxspec} R$ *is homeomorphic to* $\operatorname{P.prim} R$.

Proof. Observe first that the map $\mathcal{P}(-)$ is continuous. It is surjective by definition of $\operatorname{P.prim} R$.

We claim that $P = \bigcap \{\mathfrak{m} \in \operatorname{maxspec} R \mid \mathcal{P}(\mathfrak{m}) = P\}$, for any Poisson-primitive ideal P of R. Since P is locally closed in $\operatorname{P.spec} R$ (by assumption), the singleton $\{P\}$ is open in its closure $V_P(P)$, and so $\{P\} = V_P(P) \setminus V_P(J)$ for some Poisson ideal J of R. Note that $J \not\subseteq P$; hence, after replacing J by $J + P$, we may assume that $J \supsetneq P$. If $\mathfrak{m} \supseteq P$ is a maximal ideal such that $\mathcal{P}(\mathfrak{m}) \neq P$, then $\mathfrak{m} \supseteq \mathcal{P}(\mathfrak{m}) \supseteq J$. The remaining maximal ideals containing P must intersect to P by the Nullstellensatz, verifying the claim.

Now consider a set $X \subseteq \operatorname{P.prim} R$ whose inverse image under $\mathcal{P}(-)$, call it Y, is closed in $\operatorname{maxspec} R$. Thus,

$$Y = \{\mathfrak{m} \in \operatorname{maxspec} R \mid \mathcal{P}(\mathfrak{m}) \in X\} = \{\mathfrak{m} \in \operatorname{maxspec} R \mid \mathfrak{m} \supseteq I\}$$

for some ideal I of R. If $P \in X$ and $\mathfrak{m} \in \operatorname{maxspec} R$ with $\mathcal{P}(\mathfrak{m}) = P$, then $\mathfrak{m} \in Y$, and so $\mathfrak{m} \supseteq I$. By the claim above, the intersection of these maximal ideals equals P, and thus $P \supseteq I$. Conversely, if $P \in \operatorname{P.prim} R$ and $P \supseteq I$, then $P = \mathcal{P}(\mathfrak{m})$ for some maximal ideal $\mathfrak{m} \supseteq I$, whence $\mathfrak{m} \in Y$ and $P \in X$. Therefore $X = \{P \in \operatorname{P.prim} R \mid P \supseteq I\}$, a closed set in $\operatorname{P.prim} R$. This proves that the topology on $\operatorname{P.prim} R$ is the quotient topology inherited from $\operatorname{maxspec} R$ via $\mathcal{P}(-)$.

The final statement of the lemma follows directly. □

9.4. Lemma. *Let A be a noetherian k-algebra and R a commutative noetherian Poisson k-algebra, with* $\operatorname{char} k = 0$.

(a) *A bijection $\phi : \operatorname{spec} A \to \operatorname{P.spec} R$ is a homeomorphism if and only if ϕ and ϕ^{-1} preserve inclusions.*

(b) *Assume that A is a Jacobson ring and R an affine k-algebra. Then any homeomorphism* $\operatorname{prim} A \to \operatorname{P.prim} R$ *extends uniquely to a homeomorphism* $\operatorname{spec} A \to \operatorname{P.spec} R$.

(c) *Assume that A satisfies the Dixmier-Moeglin equivalence and R the Poisson Dixmier-Moeglin equivalence. Then any homeomorphism* $\operatorname{spec} A \to \mathrm{P.spec}\, R$ *restricts to a homeomorphism* $\operatorname{prim} A \to \mathrm{P.prim}\, R$.

Proof. (a) For $P, Q \in \operatorname{spec} A$, we have $P \subseteq Q$ if and only if $Q \in \overline{\{P\}}$, and similarly in $\mathrm{P.spec}\, R$. Hence, any homeomorphism between these spaces must preserve inclusions.

Conversely, if ϕ and ϕ^{-1} preserve inclusions, then $\phi(V(P)) = V_P(\phi(P))$ for all $P \in \operatorname{spec} A$. Since the closed sets in $\operatorname{spec} A$ are exactly the finite unions of $V(P)$s (recall the proof of Lemma 8.8(a)), it follows that ϕ sends closed sets to closed sets, i.e., ϕ^{-1} is continuous. Similarly, ϕ is continuous, and hence a homeomorphism.

(b) Lemmas 8.8 and 8.9.

(c) Under the assumed equivalences, $\operatorname{prim} A$ consists of the locally closed points in $\operatorname{spec} A$, and $\mathrm{P.prim}\, R$ consists of the locally closed points in $\mathrm{P.spec}\, R$. $\square$

9.5. Example. Let $A_q = \mathcal{O}_q(k^2)$, where $k = \overline{k}$, $\operatorname{char} k = 0$, and $q \in k^\times$. View $R = \mathcal{O}(k^2)$ as the semiclassical limit of the family $(A_q)_{q \in k^\times}$, with the Poisson structure exhibited in Example 2.2(a). The torus $H = (k^\times)^2$ acts on A_q via algebra automorphisms and on R via Poisson automorphisms so that (in both cases) $(\alpha_1, \alpha_2).x_i = \alpha_i x_i$ for $(\alpha_1, \alpha_2) \in H$ and $i = 1, 2$.

Assume that q is not a root of unity. As is easily checked (e.g., [**4**, Example II.1.2]), the prime ideals of A_q are

($\diamondsuit$)
- the maximal ideals $\langle x_1 - \alpha,\, x_2 \rangle$ and $\langle x_1,\, x_2 - \beta \rangle$, for $\alpha, \beta \in k$;
- the height 1 primes $\langle x_1 \rangle$ and $\langle x_2 \rangle$;
- the zero ideal.

All of these prime ideals, except for $\langle x_1 \rangle$ and $\langle x_2 \rangle$, are primitive [**4**, Example II.7.2]. The closed sets in $\operatorname{spec} A_q$ are easily found, but we shall not list them here – see [**4**, Example II.1.2 and Exercise II.1.C].

With very similar computations, one finds the Poisson prime and Poisson-primitive ideals in R, and a list of the closed subsets of $\mathrm{P.spec}\, R$. In terms of notation, the answers are the same as for A_q – the list ($\diamondsuit$) also describes the Poisson prime ideals of R, and all of these ideals, except for $\langle x_1 \rangle$ and $\langle x_2 \rangle$, are Poisson-primitive. We conclude that there exist compatible homeomorphisms $\operatorname{prim} A_q \to \mathrm{P.prim}\, R$ and $\operatorname{spec} A_q \to \mathrm{P.spec}\, R$, sending the ideal $\langle x_1 - \alpha,\, x_2 \rangle$ of A_q to the ideal $\langle x_1 - \alpha,\, x_2 \rangle$ of R, and so on. (We say that these maps are given by "preservation of notation".) These homeomorphisms are equivariant with respect to the actions of H described above.

By inspection, all Poisson-primitive ideals of R are locally closed in $\mathrm{P.spec}\, R$. Consequently, we conclude from Lemma 9.3 that the space of symplectic cores in $\operatorname{maxspec} R \approx k^2$ is homeomorphic to $\mathrm{P.prim}\, R$.

Analyzing the prime ideals in a quantized coordinate ring typically involves investigating localizations of factor algebras, which often turn out to be quantum tori. We sketch some basic procedures used to determine prime ideals in quantum tori, and similar ones for the analogous "Poisson tori".

9.6. Some computational tools. **(a)** A *quantum torus* over k is the localization of a quantum affine space $\mathcal{O}_{\boldsymbol{q}}(k^n)$ obtained by inverting the generators x_i, that is, an algebra

$$\mathcal{O}_{\boldsymbol{q}}((k^\times)^n) = k\langle x_1^{\pm 1}, \dots, x_n^{\pm 1} \mid x_i x_j = q_{ij} x_j x_i \text{ for all } i, j \rangle,$$

where $\boldsymbol{q} = (q_{ij})$ is a multiplicatively antisymmetric $n \times n$ matrix over k. Set $T = \mathcal{O}_{\boldsymbol{q}}((k^\times)^n)$.

Since T is a $\mathbb{Z}^n$-graded algebra, with 1-dimensional homogeneous components, its center is spanned by central monomials [**21**, Lemma 1.1]. The latter are easily computed: a monomial $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$ is central if and only if $\prod_{j=1}^n q_{ij}^{m_j} = 1$ for all i. All ideals of T are induced from ideals of $Z(T)$ [**21**, Theorem 1.2; **15**, Proposition 1.4], from which it follows that contraction and extension give inverse homeomorphisms between $\operatorname{spec} T$ and $\operatorname{spec} Z(T)$ [**15**, Corollary 1.5(b)]. In particular, it follows from the above facts that T is a simple algebra if and only if $Z(T) = k$ [**36**, Proposition 1.3].

(b) Let $R = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial ring, equipped with a Poisson bracket such that $\{x_i, x_j\} = \pi_{ij} x_i x_j$ for all i, j, where (π_{ij}) is an antisymmetric $n \times n$ matrix over k. The results of part (a) all have Poisson analogs for R, as follows.

The *Poisson center* of R, denoted $Z_P(R)$, is the subalgebra consisting of those $r \in R$ for which the derivation $\{r, -\}$ vanishes. Since the Poisson bracket on R respects the $\mathbb{Z}^n$-grading, $Z_P(R)$ is spanned by the monomials it contains [**21**, Lemma 2.1; **52**, Lemma 1.2(a)]. A monomial $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$ is Poisson central if and only if $\sum_{j=1}^n \pi_{ij} m_j = 0$ for all i. All Poisson ideals of R are induced from ideals of $Z_P(R)$ [**21**, Theorem 2.2; **52**, Lemma 1.2(b)], from which it follows that contraction and extension give inverse homeomorphisms between $\operatorname{P.spec} R$ and $\operatorname{spec} Z_P(R)$. In particular, it follows from the above facts that R is Poisson-simple (meaning that it has no proper nonzero Poisson ideals) if and only if $Z_P(R) = k$.

9.7. Example. Let $A_q = \mathcal{O}_q(SL_2(k))$, where $k = \overline{k}$, $\operatorname{char} k = 0$, and $q \in k^\times$. View $R = \mathcal{O}(SL_2(k))$ as the semiclassical limit of the family $(A_q)_{q \in k^\times}$, with the Poisson structure exhibited in Example 2.2(c). The torus $H = (k^\times)^2$ again acts on A_q and R, this time so that

$$\begin{aligned} (\alpha, \beta).X_{11} &= \alpha\beta X_{11} & \qquad (\alpha, \beta).X_{12} &= \alpha\beta^{-1} X_{12} \\ (\alpha, \beta).X_{21} &= \alpha^{-1}\beta X_{21} & \qquad (\alpha, \beta).X_{22} &= \alpha^{-1}\beta^{-1} X_{22} \end{aligned}$$

for $(\alpha, \beta) \in H$.

Now restrict q to a non-root of unity. The prime ideals of A_q can be computed with the tools of §9.6(a), as outlined in [**4**, Exercise II.1.D]. For instance, one checks that A_q has a localization

$$A_q[X_{11}^{-1}, X_{12}^{-1}, X_{21}^{-1}] \cong k\langle x^{\pm 1}, y^{\pm 1}, z^{\pm 1} \mid xy = qyx,\ xz = qzx,\ yz = zy \rangle,$$

and that the center of the latter algebra is $k[(yz^{-1})^{\pm 1}]$. It follows that the prime ideals of A_q not containing X_{12} or X_{21} consist of $\langle 0 \rangle$ and $\langle X_{12} - \lambda X_{21} \rangle$, for $\lambda \in k^\times$. The full list of prime ideals of A_q is as follows:

- the maximal ideals $\langle X_{11} - \lambda,\ X_{12},\ X_{21},\ X_{22} - \lambda^{-1} \rangle$, for $\lambda \in k^\times$;

(♠) • the ideal $\langle X_{12}, X_{21}\rangle$;
• the height 1 primes $\langle X_{21}\rangle$ and $\langle X_{12} - \lambda X_{21}\rangle$, for $\lambda \in k$;
• the zero ideal.

A diagram of spec A_q, with inclusions marked, is given in [**4**, Diagram II.1.3].

A similar computation, using §9.6(b), yields the Poisson prime ideals of R, which can be described exactly as in (♠). This provides a natural H-equivariant bijection $\phi : \operatorname{spec} A_q \to \text{P.spec}\, R$, given by "preservation of notation". By inspection, ϕ and ϕ^{-1} preserve inclusions, and thus, by Lemma 9.4(a), ϕ is a homeomorphism.

The algebra A_q satisfies the Dixmier-Moeglin equivalence by [**4**, Corollary II.8.5], and R satisfies the Poisson Dixmier-Moeglin equivalence by [**13**, Theorem 4.3]. Therefore Lemma 9.4(c) implies that ϕ restricts to a homeomorphism $\operatorname{prim} A_q \to \text{P.prim}\, R$. In A_q, all prime ideals are primitive except for $\langle X_{12}, X_{21}\rangle$ and $\langle 0\rangle$ (cf. [**4**, Example II.8.6]). Similarly, in R all Poisson prime ideals are Poisson-primitive except for $\langle X_{12}, X_{21}\rangle$ and $\langle 0\rangle$. As in the previous example, we can use Lemma 9.3 to see that the space of symplectic cores in $\operatorname{maxspec} R \approx SL_2(k)$ is homeomorphic to $\text{P.prim}\, R$.

9.8. Evidence for Conjecture 9.1. In most of the instances discussed below, k is assumed to be algebraically closed of characteristic zero.

(a) Examples 9.5 and 9.7 are the most basic instances in which the conjecture has been verified. In the same way (although with somewhat more effort), one can verify it for $\mathcal{O}_q(GL_2(k))$. In particular, most of the work required to determine the prime ideals in the generic $\mathcal{O}_q(GL_2(k))$ is done in [**4**, Example II.8.7].

(b) We next turn to the quantized coordinate rings $\mathcal{O}_q(G)$ and $\mathcal{O}_{q,p}(G)$ over $k = \mathbb{C}$, where G is a connected semisimple complex Lie group, $q \in k^\times$ is not a root of unity, and p is an antisymmetric bicharacter on the weight lattice of G (as in [**24**, §3.4]).

The Poisson structure on $\mathcal{O}(G)$ resulting from the semiclassical limit process gives G the combined structure of a *Poisson-Lie group* (e.g., see [**33**, Chapter 1] for the concept, and [**22**, §A.1] for the result). There is a known recipe for the symplectic leaves in G in case the Poisson structure arises from the standard quantization [**22**, Appendix A], and similarly in the multiparameter "algebraic" case [**24**, Theorem 1.8]. In both these cases, it follows that the symplectic leaves are Zariski locally closed (see [**5**, Theorem 1.9] for a more explicit statement). Hence, the symplectic leaves in G coincide with the symplectic cores (Theorem 7.1(b)).

As discussed in §4.4, Hodges and Levasseur put forward the conjecture that there should be a bijection between $\operatorname{prim} \mathcal{O}_q(G)$ and the set of symplectic leaves in G [**22**, §2.8, Conjecture 1]. They developed such bijections for $G = SL_n(\mathbb{C})$ in [**23**], and for general G in their work with Toro [**24**]. More generally, Hodges, Levasseur, and Toro established a bijection between $\operatorname{prim} \mathcal{O}_{q,p}(G)$ and the set of symplectic leaves in G in the algebraic case. All these bijections are equivariant with respect to natural actions of a maximal torus of G.

Except for the case $G = SL_2(\mathbb{C})$ covered in Example 9.7, the topological properties of the above bijections are not known. Even when $G = SL_3(\mathbb{C})$, it is not known whether $\operatorname{prim} \mathcal{O}_q(G)$ is homeomorphic to the space of symplectic leaves (= cores) in G.

(c) The prime and primitive spectra of general multiparameter quantum affine spaces $\mathcal{O}_{\boldsymbol{q}}(k^n)$ were analyzed by Goodearl and Letzter in [**17**], assuming $k = \overline{k}$ together with a mi-

nor technical assumption (that either $\operatorname{char} k = 2$, or -1 is not in the subgroup $\langle q_{ij} \rangle \subseteq k^\times$). They proved that there are compatible topological quotient maps $k^n \approx \operatorname{maxspec} \mathcal{O}(k^n) \to \operatorname{prim} \mathcal{O}_{\boldsymbol{q}}(k^n)$ and $\operatorname{spec} \mathcal{O}(k^n) \to \operatorname{spec} \mathcal{O}_{\boldsymbol{q}}(k^n)$, equivariant with respect to natural actions of the torus $(k^\times)^n$ [**17**, Theorem 4.11]. Similar results were proved not only for quantum tori $\mathcal{O}_{\boldsymbol{q}}((k^\times)^n)$ [**17**, Theorem 3.11] but also for quantum affine toric varieties [**17**, Theorem 6.3].

Oh, Park, and Shin converted these topological quotient results into the following (assuming $\operatorname{char} k = 0$ and $-1 \notin \langle q_{ij} \rangle \subseteq k^\times$): For each $\mathcal{O}_{\boldsymbol{q}}(k^n)$, there is a Poisson structure on $\mathcal{O}(k^n)$ such that there are compatible homeomorphisms $\operatorname{P.prim} \mathcal{O}(k^n) \to \operatorname{prim} \mathcal{O}_{\boldsymbol{q}}(k^n)$ and $\operatorname{P.spec} \mathcal{O}(k^n) \to \operatorname{spec} \mathcal{O}_{\boldsymbol{q}}(k^n)$ [**41**, Theorem 3.5]. Goodearl and Letzter, finally, showed that such homeomorphisms could be obtained for semiclassical limit Poisson structures [**18**, Theorem 3.6], and extended the results to quantum affine toric varieties [**18**, Theorem 5.2].

All these Poisson algebra structures on $\mathcal{O}(k^n)$ satisfy the Poisson Dixmier-Moeglin equivalence [**13**, Example 4.6]. Hence, the space of symplectic cores in k^n is homeomorphic to $\operatorname{prim} \mathcal{O}_{\boldsymbol{q}}(k^n)$, via Lemma 9.3. The symplectic cores in k^n are algebraic, whereas this does not always hold for the symplectic leaves, as shown by Vancliff [**52**, Corollary 3.4]. An explicit example is computed in [**18**, Example 3.10].

(d) The prime and primitive spectra of the algebras $K^{P,Q}_{n,\Gamma}(k)$ introduced by Horton [**25**] were analyzed by Oh in [**40**]. These algebras are multiparameter quantizations of $\mathcal{O}(k^{2n})$, and include quantum symplectic spaces $\mathcal{O}_q(\mathfrak{sp}\, k^{2n})$, even-dimensional quantum euclidean spaces $\mathcal{O}_q(\mathfrak{o}\, k^{2n})$, and quantum Heisenberg spaces, among others. Oh introduced Poisson algebra structures $A^{P,Q}_{n,\Gamma}(k)$ on $\mathcal{O}(k^{2n})$, and constructed compatible homeomorphisms $\operatorname{P.prim} A^{P,Q}_{n,\Gamma}(k) \to \operatorname{prim} K^{P,Q}_{n,\Gamma}(k)$ and $\operatorname{P.spec} A^{P,Q}_{n,\Gamma}(k) \to \operatorname{spec} K^{P,Q}_{n,\Gamma}(k)$, assuming the parameters involved in P, Q, Γ are suitably generic [**40**, Theorem 4.14].

As stated in Remark 9.2(b), a quantized coordinate ring may belong to some flat families for which Conjecture 9.1 holds and also to others for which it fails. We outline Vancliff's example [**52**, Example 3.14] illustrating this phenomenon.

9.9. Example. **(a)** Let $a_i = i - 1$ for $i = 1, 2, 3$, and set

$$\begin{aligned} R_0 &= \mathbb{C}[h][(1 + a_i h)^{-1} \mid i = 1, 2, 3] \\ A &= R_0\langle x_1, x_2, x_3 \mid x_i x_j = r_{ij} x_j x_i \text{ for } i, j = 1, 2, 3\rangle, \end{aligned}$$

where $r_{ij} = (1 + a_i h)(1 + a_j h)^{-1}$ for all i, j. This defines a flat family of $\mathbb{C}$-algebras, whose semiclassical limit is the polynomial ring $R = \mathbb{C}[x_1, x_2, x_3]$ with the Poisson bracket satisfying

$$\{x_1, x_2\} = -x_1 x_2 \qquad \{x_1, x_3\} = -2x_1 x_3 \qquad \{x_2, x_3\} = -x_2 x_3\,.$$

It follows from [**52**, Corollary 3.4] that the symplectic leaves in $\mathbb{C}^3$ for this Poisson structure are algebraic; hence, they coincide with the symplectic cores (Theorem 7.1(b)). By [**13**,

Example 4.6], R satisfies the Poisson Dixmier-Moeglin equivalence, and so Lemma 9.3 implies that the space of symplectic leaves in $\mathbb{C}^3$ is homeomorphic to $\text{P.prim}\, R$.

The Poisson-primitive ideals of R are listed in [**52**, Example 3.14] (where they are labelled "maximal Poisson ideals"). They consist of

($\diamondsuit$) • the maximal ideals $\langle x_1 - \alpha, x_2, x_3\rangle$, $\langle x_1, x_2 - \beta, x_3\rangle$, $\langle x_1, x_2, x_3 - \gamma\rangle$, for $\alpha, \beta, \gamma \in \mathbb{C}$;
• the height 1 primes $\langle x_1\rangle$, $\langle x_2\rangle$, $\langle x_3\rangle$, and $\langle x_1x_3 - \lambda x_2^2\rangle$, for $\lambda \in \mathbb{C}^\times$.

We can compute them by using §9.6(b) to find the Poisson prime ideals of R and then applying the Poisson Dixmier-Moeglin equivalence. For instance, the Poisson center of the localization $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$ is $\mathbb{C}[(x_1x_2^{-2}x_3)^{\pm 1}]$, from which it follows that any nonzero Poisson prime ideal of R must contain either one of the x_i or else $x_1x_3 - \lambda x_2^2$ for some $\lambda \in \mathbb{C}^\times$. The full list of Poisson prime ideals of R is

- $\langle x_1 - \alpha, x_2, x_3\rangle$, $\langle x_1, x_2 - \beta, x_3\rangle$, $\langle x_1, x_2, x_3 - \gamma\rangle$, for $\alpha, \beta, \gamma \in \mathbb{C}$;
- $\langle x_1, x_2\rangle$, $\langle x_1, x_3\rangle$, $\langle x_2, x_3\rangle$;
- $\langle x_1\rangle$, $\langle x_2\rangle$, $\langle x_3\rangle$, $\langle x_1x_3 - \lambda x_2^2\rangle$, for $\lambda \in \mathbb{C}^\times$;
- $\langle 0\rangle$.

Inspection immediately shows that the locally closed points of $\text{P.spec}\, R$ are the Poisson prime ideals listed in ($\diamondsuit$).

(b) A generic member of the flat family given by A is $B_q = A/A(h - q)A$, where q is a complex scalar such that $1 + q$ and $1 + 2q$ generate a free abelian subgroup of rank 2 in $\mathbb{C}^\times$. For such q, the primitive ideals of B_q, as stated in [**52**, Example 3.14], consist of

- $\langle x_1 - \alpha, x_2, x_3\rangle$, $\langle x_1, x_2 - \beta, x_3\rangle$, $\langle x_1, x_2, x_3 - \gamma\rangle$, for $\alpha, \beta, \gamma \in \mathbb{C}$;
- $\langle x_1\rangle$, $\langle x_2\rangle$, $\langle x_3\rangle$, $\langle 0\rangle$.

These can be computed by finding the prime ideals using §9.6(a) and then applying the Dixmier-Moeglin equivalence, which holds because B_q is a quantum affine space [**15**, Corollary 2.5].

Observe that $\text{prim}\, B_q$ is not homeomorphic to $\text{P.prim}\, R$. For instance, $\text{prim}\, B_q$ has a generic point, while $\text{P.prim}\, R$ does not.

(c) In contrast to the above, any generic B_q is a member of a flat family with a semiclassical limit R' (the algebra R, but with a different Poisson structure) such that $\text{prim}\, B_q \approx \text{P.prim}\, R'$ and $\text{spec}\, B_q \approx \text{P.spec}\, R'$, by [**18**, Theorem 3.6].

It would be very interesting to obtain criteria to determine which flat families yield "good" semiclassical limits relative to Conjecture 9.1. For quantum affine spaces and their Poisson analogs, one good condition appears in the work of Oh, Park, and Shin [**41**, Theorem 3.5] – roughly, if the scalars appearing in the defining Poisson brackets of the semiclassical limit arise from an embedding into k^+ of the subgroup of $k^\times$ generated by the scalars appearing in the defining commutation relations of the quantum affine space, then the prime and primitive spectra of the quantum affine space are homeomorphic to the Poisson prime and Poisson-primitive spectra of the semiclassical limit.

We close with an example of the "simplest possible" quantum group for which primitive ideals match symplectic cores but not symplectic leaves. There is no nontrivial multiparameter version of quantum SL_2, and to deal with $\mathcal{O}_{q,p}(SL_3(\mathbb{C}))$ would require investigating 36 families of primitive ideals (indexed by $S_3 \times S_3$, as in [**24**, Corollary 4.5]). Instead, we

look at a multiparameter quantization of GL_2. It is convenient to use Takeuchi's original presentation [**49**].

9.10. Example. For the classification of primitive ideals, we assume only that k is algebraically closed, and we choose a generic pair of parameters $p, q \in k^\times$, meaning that they generate a free abelian subgroup of rank 2 in $k^\times$. We restrict to $k = \mathbb{C}$ and special choices of p and q when setting up a semiclassical limit and discussing symplectic leaves.

(a) Define the two-parameter quantum 2×2 matrix algebra $M_{q^{-1},p}$ as in [**49**]. This is the k-algebra with generators X_{11}, X_{12}, X_{21}, X_{22} and relations

$$\begin{aligned} X_{11}X_{12} &= qX_{12}X_{11} & X_{11}X_{21} &= p^{-1}X_{21}X_{11} \\ X_{21}X_{22} &= qX_{22}X_{21} & X_{12}X_{22} &= p^{-1}X_{22}X_{12} \\ X_{12}X_{21} &= (pq)^{-1}X_{21}X_{12} & X_{11}X_{22} - X_{22}X_{11} &= (q-p)X_{12}X_{21}\,. \end{aligned}$$

The element $D = X_{11}X_{22} - qX_{12}X_{21}$ is the quantum determinant in $M_{q^{-1},p}$, but it is normal rather than central:

$$X_{ij}D = (pq)^{i-j}DX_{ij}$$

for all i, j [**49**, §2]. Since the powers of D form an Ore set, we can construct the Ore localization $A = A_{q^{-1},p} = M_{q^{-1},p}[D^{-1}]$. There is a Hopf algebra structure on A [**49**, §2], but we do not need that here.

For comparison with other presentations of multiparameter quantized coordinate rings, we point out that $A = \mathcal{O}_{pq^{-1},\boldsymbol{p}}(GL_2(k))$ in the notation of [**12**, §1.3; **4**, §I.2.4]), where $\boldsymbol{p} = \left[\begin{smallmatrix} 1 & q^{-1} \\ q & 1 \end{smallmatrix}\right]$. In particular, [**4**, Corollary II.6.10] applies, implying that all prime ideals of A are completely prime.

Observe that X_{12} and X_{21} are normal in A, and so we can localize with respect to their powers. Although X_{11} is not normal, its powers also form an Ore set (e.g., verify this first in $M_{q^{-1},p}$, which is an iterated skew polynomial ring over $k[X_{11}]$). Note that any ideal I of A which contains X_{11} also contains $X_{12}X_{21}$, whence $D \in I$ and $I = A$. Hence, no prime ideal of A contains X_{11}, which means that no prime ideals of A are lost in passing from A to the localization $A[X_{11}^{-1}]$.

(b) The quotient $A/\langle X_{12}, X_{21}\rangle$ is isomorphic to a commutative Laurent polynomial ring $k[x_{11}^{\pm1}, x_{22}^{\pm1}]$. Hence, we know the prime ideals of A containing $\langle X_{12}, X_{21}\rangle$. The others correspond to prime ideals in the localizations $\big(A/\langle X_{12}\rangle\big)[X_{21}^{-1}]$, $\big(A/\langle X_{21}\rangle\big)[X_{12}^{-1}]$, and $A[X_{11}^{-1}, X_{12}^{-1}, X_{21}^{-1}]$. We claim that these localizations are simple algebras, from which it will follow that the only prime ideals of A not containing $\langle X_{12}, X_{21}\rangle$ are $\langle X_{12}\rangle$, $\langle X_{21}\rangle$, and $\langle 0\rangle$.

First, $\big(A/\langle X_{12}\rangle\big)[X_{21}^{-1}]$ is isomorphic to the algebra

$$T_1 := k\langle x^{\pm1}, y^{\pm1}, z^{\pm1} \mid xy = p^{-1}yx,\ xz = xz,\ yz = qzy\rangle.$$

Via §9.6(a), we compute that $Z(T_1) = k$, whence T_1 is simple. Similarly, $\big(A/\langle X_{21}\rangle\big)[X_{12}^{-1}]$ is simple.

Third, observe that $X_{22} = X_{11}^{-1}(D + qX_{12}X_{21})$ in $A[X_{11}^{-1}]$, and so this algebra can be generated by $X_{11}^{\pm 1}$, X_{12}, X_{21}, $D^{\pm 1}$. Consequently, $A[X_{11}^{-1}, X_{12}^{-1}, X_{21}^{-1}]$ is isomorphic to the k-algebra T_3 with generators $x^{\pm 1}$, $y^{\pm 1}$, $z^{\pm 1}$, $w^{\pm 1}$ and relations

$$
\begin{aligned}
&xy = qyx && xz = p^{-1}zx && xw = wx \\
&yz = (pq)^{-1}zy && yw = (pq)^{-1}wy && zw = pqwz.
\end{aligned}
$$

Another application of §9.6(a) shows that T_3 is simple, establishing the claim.

Therefore, the prime ideals of A consist of

- the maximal ideals $\langle X_{11} - \lambda,\ X_{12},\ X_{21},\ X_{22} - \mu\rangle$, for $\lambda, \mu \in k^\times$;
- ($\diamondsuit$) the ideals $\langle X_{12},\ X_{21},\ f(X_{11}, X_{22})\rangle$, for irreducible polynomials $f(s,t) \in k[s^{\pm 1}, t^{\pm 1}]$;
- the ideals $\langle X_{12},\ X_{21}\rangle$, $\langle X_{12}\rangle$, $\langle X_{21}\rangle$, and $\langle 0\rangle$.

(c) The torus $H = (k^\times)^4$ acts on A by k-algebra automorphisms such that

$$
(\alpha_1, \alpha_2, \beta_1, \beta_2).X_{ij} = \alpha_i \beta_j X_{ij} \tag{9.10c}
$$

for all i, j. Only four of the prime ideals of A are H-stable, and thus [**16**, Corollary 2.7(ii), Remark 5.9(i)] implies that A satisfies the Dixmier-Moeglin equivalence (cf. [**4**, Corollary II.8.5(c)]). Therefore, the primitive ideals of A are

- the maximal ideals $\langle X_{11} - \lambda,\ X_{12},\ X_{21},\ X_{22} - \mu\rangle$, for $\lambda, \mu \in k^\times$;
- the ideals $\langle X_{12}\rangle$, $\langle X_{21}\rangle$, and $\langle 0\rangle$.

(d) Now restrict to $k = \mathbb{C}$, choose $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, assume that q is transcendental over $\mathbb{Q}(\alpha)$, and take $p = 1 + \alpha(q-1)$. The assumptions on α and q ensure that the subgroup $\langle p, q\rangle \subseteq \mathbb{C}^\times$ is free abelian of rank 2, as needed above. Our choice of p is a first-order Taylor approximation of q^α, which is convenient for extension to polynomial rings.

Choose a Laurent polynomial ring $k[z^{\pm 1}]$, set $z_\alpha = 1 + \alpha(z-1)$, and let $B = M_{z^{-1}, z_\alpha}$ over $k[z^{\pm 1}, z_\alpha^{-1}]$ in the notation of [**49**, §2]. Thus, B is the $k[z^{\pm 1}, z_\alpha^{-1}]$-algebra given by generators X_{11}, X_{12}, X_{21}, X_{22} and relations

$$
\begin{aligned}
&X_{11}X_{12} = zX_{12}X_{11} && X_{11}X_{21} = z_\alpha^{-1}X_{21}X_{11} \\
&X_{21}X_{22} = zX_{22}X_{21} && X_{12}X_{22} = z_\alpha^{-1}X_{22}X_{12} \\
&X_{12}X_{21} = (zz_\alpha)^{-1}X_{21}X_{12} && X_{11}X_{22} - X_{22}X_{11} = (z - z_\alpha)X_{12}X_{21}\,.
\end{aligned}
$$

Observe that B is an iterated skew polynomial algebra over $k[z^{\pm 1}, z_\alpha^{-1}]$, and so it is torsionfree over $k[z^{\pm 1}]$. This algebra has been arranged so that $B/(z-q)B \cong M_{q^{-1},p}$ and $B/(z-1)B \cong \mathcal{O}(M_2(k))$. In B, the quantum determinant is $D = X_{11}X_{22} - zX_{12}X_{21}$, and it is normal. We set $C = B[D^{-1}]$ and observe that C is a torsionfree $k[z^{\pm 1}]$-algebra such that $C/(z-q)C \cong A$ and $C/(z-1)C \cong R := \mathcal{O}(GL_2(k))$.

Thus, A is one of the quantizations of R in the family of algebras $C/(z-\gamma)C$. The semiclassical limit of this family is the algebra R, equipped with the Poisson bracket

determined by

$$\begin{aligned} \{X_{11}, X_{12}\} &= X_{11}X_{12} & \{X_{11}, X_{21}\} &= -\alpha X_{11}X_{21} \\ \{X_{21}, X_{22}\} &= X_{21}X_{22} & \{X_{12}, X_{22}\} &= -\alpha X_{12}X_{22} \\ \{X_{12}, X_{21}\} &= -(1+\alpha)X_{12}X_{21} & \{X_{11}, X_{22}\} &= (1-\alpha)X_{12}X_{21}\,. \end{aligned}$$

To find the Poisson prime ideals of R, we can proceed in parallel with part (b) above, using §9.6(b) in place of §9.6(a). We compute that the Poisson prime ideals of R can be listed exactly as in ($\diamondsuit$). This yields an obvious bijection $\phi : \operatorname{spec} A \to \text{P.spec}\, R$ given by "preservation of notation". Clearly ϕ and ϕ^{-1} preserve inclusions, and so ϕ is a homeomorphism by Lemma 9.4(a). (Alternatively, one can easily identify the closed sets in $\operatorname{spec} A$ and $\text{P.spec}\, R$ and then check that ϕ and ϕ^{-1} are closed maps.)

The torus H acts on R by Poisson algebra automorphisms satisfying (9.10c), and only four Poisson prime ideals of R are stable under this action. Consequently, [**13**, Theorem 4.3] implies that R satisfies the Poisson Dixmier-Moeglin equivalence. Thus, the Poisson-primitive ideals of R are

($\spadesuit$)
- the maximal ideals $\langle X_{11} - \lambda,\, X_{12},\, X_{21},\, X_{22} - \mu\rangle$, for $\lambda, \mu \in k^\times$;
- the ideals $\langle X_{12}\rangle$, $\langle X_{21}\rangle$, and $\langle 0\rangle$,

and therefore ϕ restricts to a homeomorphism $\operatorname{prim} A \to \text{P.prim}\, R$.

(e) In view of ($\spadesuit$), we can now identify the symplectic cores in $GL_2(\mathbb{C}) \approx \operatorname{maxspec} R$ with respect to the Poisson structure under discussion. They are

- the singletons $\left\{\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}\right\}$, for $\lambda, \mu \in \mathbb{C}^\times$;
- the sets $\begin{bmatrix} \mathbb{C}^\times & \mathbb{C}^\times \\ 0 & \mathbb{C}^\times \end{bmatrix}$, $\begin{bmatrix} \mathbb{C}^\times & 0 \\ \mathbb{C}^\times & \mathbb{C}^\times \end{bmatrix}$ and $\left\{\begin{bmatrix} \lambda & \beta \\ \gamma & \mu \end{bmatrix} \in GL_2(\mathbb{C}) \;\middle|\; \beta, \gamma \neq 0\right\}$.

The space of symplectic cores in $GL_2(\mathbb{C})$ is homeomorphic to $\text{P.prim}\, R$ by Lemma 9.3.

Since $\begin{bmatrix} \mathbb{C}^\times & \mathbb{C}^\times \\ 0 & \mathbb{C}^\times \end{bmatrix}$ and $\begin{bmatrix} \mathbb{C}^\times & 0 \\ \mathbb{C}^\times & \mathbb{C}^\times \end{bmatrix}$ are complex manifolds of odd dimension, they cannot be symplectic leaves. In fact, each is the union of a one-parameter family of symplectic leaves, which can be calculated as in [**18**, Example 3.10(v)]. For instance, the symplectic leaves contained in $\begin{bmatrix} \mathbb{C}^\times & \mathbb{C}^\times \\ 0 & \mathbb{C}^\times \end{bmatrix}$ are the surfaces

$$\left\{\begin{bmatrix} \lambda & \beta \\ 0 & \delta\lambda^\alpha \end{bmatrix} \;\middle|\; \lambda, \beta \in \mathbb{C}^\times\right\},$$

for $\delta \in \mathbb{C}^\times$.

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